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The von Neumann algebra characterization theorems

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Dedicated to the memory of a friend and colleague, Dock Sang Rim

1. Introduction

When von Neumann introduced the class of operator algebras that have become known as "von Neumann algebras", he defined them in terms of a representation on a Hilbert space [11]. One of the early puzzles of the subject, and one that intrigued von Neumann [12], was the question of a representation-independent characterization of such algebras. Is there a (sensible) set of conditions one can impose on a Banach algebra that causes it to be isomorphic to a von Neumann algebra?

The first efforts to solve this problem were plagued by a lack of suitably developed technique. Without the clear understanding of the representation-independent nature of C^*-algebras provided by the pioneering work of Gelfand-Neumark [5] and Segal [17], the difficulties in solving the von Neumann algebra problem are multiplied manyfold. It is not easy to recognize such difficulties from the dizzying technical heights that the modern theory of operator algebras has attained, but the development of mathematics replays the program of "unapproachable problem, acclaimed solution, forgotten difficulties" often enough (and with fresh enough material) that most of us come to understand its nature.

With [5, 17] as background and with the sharpened techniques that had begun to develop by the mid 1950's, the von Neumann algebra characterization problem assumed manageable proportions. Both of the standard characterizations [7, 16] were established and appeared virtually simultaneously and (totally) independently – which must be some measure of the "ripeness" of the problem for solution. The characterization in [16] stems from Dixmier's proof [3] that each von Neumann algebra is the dual (conjugate) space to its (Banach) space of ultraweakly continuous functionals. (Definitions and details will appear in later sections.) It follows the lead of Takeda [21] and characterizes the von Neumann algebras among C^*-algebras as those that are dual spaces.

Certain (algebraic) order properties of special C^*-algebras were known to single out a subclass with properties remarkably like those of the von Neumann al-

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gebras [9, 15]. It was also known (by commutative example [4]) that not all such C^*-algebras are von Neumann algebras. M.H. Stone's fundamental analysis of the commutative case (in the form of algebras of continuous functions) [18, 19, 20] was augmented by Dixmier in [4] where he describes those function algebras that are isomorphic to von Neumann algebras. Following the basic principle of the subject of operator algebras: the theory of C^*-algebras is non-commutative continuous-function theory and the theory of von Neumann algebras is non-commutative measurable-function theory, the result of [4] transcribed to the non-commutative case becomes a conjecture whose affirmation characterizes von Neumann algebras. The conjecture was not made explicit because it seemed at that time to be more a "half-hope" that a probable result. That result is the characterization established in [7].

The virtue of a characterization result may lie in its providing a vantage point from which to develop a subject. Such is the case with the Gelfand-Neumark characterization of C^*-algebras [5]. It is also the case with Sakai's characterization of von Neumann algebras [16]. This latter is succinct and easily grasped in an initial study of the subject. The virtue of a characterization result may lie in the methods it introduces. Segal's version [17] of [5] introduces the state-representation technique basic to C^*-algebra theory. In [7], the non-commutative monotone-order techniques are introduced; they culminate in Pedersen's beautiful "up-down" theorem [13] and his equally splendid [14]. The characterization itself may be a tool of special significance as is the case with Connes's characterization of *matricial* von Neumann algebras [2].

In the sections that follow, we produce a complete and readily accessible account of the von Neumann algebra characterization results. The third section contains a description of the commutative case including an example of a function algebra with the "algebraic" properties of a (commutative) von Neumann algebra that is not such an algebra. In the fourth section, the non-commutative characterization results appear. The arguments of [7] are improved somewhat. The Sakai characterization [16] is derived from [7], which helps to clarify the interrelation between [16] and [7] that has not been explicit before. The final section contains an account of Tomiyama's proof of the Sakai characterization using the "projection-of-norm-one" and "universal representation" techniques [22, 23].

Our reference for the general theory of operator algebras is [8]. The author notes with gratitude the partial support of the National Science Foundation (USA) during the preparation of this article.

2. Preliminaries

The starting point for this account is the abstract (Banach algebra) description of C^*-algebras. In [5], it is shown that a (complex) Banach algebra $\mathfrak{A}$ with a unit I and an adjoint operation (involution) $A \to A^*$ that satisfies:

(i) $(aA+B)^*=\bar{a}A^*+B^*$
(ii) $(AB)^*=B^*A^*$
(iii) $(A^*)^*=A$
(iv) $\|A^*A\|=\|A^*\|\,\|A\|$

and the further conditions

(v) A^*A+I is invertible
(vi) $\|A^*\|=\|A\|$

is isomorphic to an algebra $\mathfrak{A}_0$ of bounded operators acting on a complex Hilbert space $\mathscr{H}$. The isomorphism ϕ is an isometry ($\|A\|=\|\phi(A)\|$, where $\|\phi(A)\|$ is the *bound* of the operator $\phi(A)$). In addition, $\phi(A^*)=\phi(A)^*$ for each A in $\mathfrak{A}$, where $\phi(A)^*$ is the operator on $\mathscr{H}$ adjoint to $\phi(A)$ — we say that ϕ is *adjoint preserving.* With variations in its application, the terminology in use describes $\mathfrak{A}$ as a "B^*-algebra" and $\mathfrak{A}_0$ as a "C^*-algebra." It results from this theorem (the "Gelfand-Neumark" theorem) that $\mathfrak{A}_0$ is a (norm) closed subalgebra of $\mathscr{B}(\mathscr{H})$, the algebra of all bounded operators on $\mathscr{H}$, and that A^* is in $\mathfrak{A}_0$ when A is in $\mathfrak{A}_0$ (we say that $\mathfrak{A}_0$ is a *self-adjoint* subalgebra of $\mathscr{B}(\mathscr{H})$ in this case).

Gelfand and Neumark conjectured that conditions (v) and (vi) are redundant; this is indeed the case. A report of that subject, together with a proof of the full result, can be found in [6] (where condition (vi) was removed).

Segal [17] proceeds from [5] to a definition of *states* of a C^*-algebra and a construction (the GNS *construction*) of *representations* of C^*-algebras engendered by states. With A in $\mathfrak{A}$, a complex number λ is said to lie in the spectrum of A (relative to $\mathfrak{A}$) when $A-\lambda I$ is not invertible in $\mathfrak{A}$ (and $\operatorname{sp} A$, the spectrum of A, is the set of such λ). If $A=A^*$ (we say that A is self-adjoint in this case) and $\operatorname{sp} A$ consists of non-negative real numbers, we say that A is *positive*. (It can be proved, though non trivially, that $\operatorname{sp} A$ consists of real numbers when A is self-adjoint.) A linear functional ϱ defined on $\mathfrak{A}$ that takes non-negative real values at positive elements of $\mathfrak{A}$ and takes the value 1 at I is said to be a *state* of $\mathfrak{A}$. The conjugate-bilinear form $(A,B)\to\varrho(B^*A)$ is a positive semi-definite inner product on $\mathfrak{A}$ when ϱ is a state of $\mathfrak{A}$. The set $\{A:\varrho(A^*A)=0\}$ of null vectors of this inner product is a closed left ideal $\mathscr{L}$ in $\mathfrak{A}$ and the conjugate-bilinear form

$$(A+\mathscr{L},B+\mathscr{L})\to\varrho(B^*A)(=\langle A+\mathscr{L},B+\mathscr{L}\rangle_\varrho)$$

is a (well-defined) positive-definite inner product on the quotient vector space $\mathfrak{A}/\mathscr{L}$. The completion $\mathscr{H}_\varrho$ of $\mathfrak{A}/\mathscr{L}$ relative to $\langle\ ,\ \rangle_\varrho$ is a Hilbert space. Each A in $\mathfrak{A}$ gives rise to a bounded operator on $\mathfrak{A}/\mathscr{L}$ (bounded relative to the norm derived from the inner product $\langle\ ,\ \rangle_\varrho$) by means of the mapping, $B+\mathscr{L}\to AB+\mathscr{L}$. There is a unique extension of this operator to a bounded operator $\pi_\varrho(A)$ on $\mathscr{H}_\varrho$.

The mapping π_ϱ of $\mathfrak{A}$ into $\mathscr{B}(\mathscr{H}_\varrho)$ is a homomorphism with the additional properties, $\pi_\varrho(A^*)=\pi_\varrho(A)^*$ and $\|\pi_\varrho(A)\|\le\|A\|$. We say that π_ϱ is a * *representation* of $\mathfrak{A}$ (*the GNS representation corresponding to* ϱ). The proof of the Gelfand-

Neumark theorem is effected by showing that the set $\mathscr{S}$ of states of $\mathfrak{A}$ separates $\mathfrak{A}$ (if A is positive and $\varrho(A)=0$ for each ϱ in $\mathscr{S}$, then $A=0$) and then forming the direct sum π of the representations π_ϱ. That is $\pi(A)y=\sum_{\varrho\in\mathscr{S}}\oplus\pi_\varrho(A)y_\varrho$, where $y=\sum_{\varrho\in\mathscr{S}}\oplus y_\varrho$ (in $\sum_{\varrho\in\mathscr{S}}\oplus\mathscr{H}_\varrho$). The fact that $\mathscr{S}$ is separating translates into the fact that π is an isomorphism – so that any other separating family could be used in place of $\mathscr{S}$.

The description of a von Neumann algebra is given first in the context of a representation on a Hilbert space $\mathscr{H}$. We employ a topology on $\mathscr{B}(\mathscr{H})$ different from the norm topology (when $\mathscr{H}$ is definite-dimensional). Any one of several (distinct) topologies would do – and the fact that each of these topologies will serve is a series of technical results important to the theory – but the simplest to describe is probably the "point-open" topology on $\mathscr{B}(\mathscr{H})$ (as a set of mappings from $\mathscr{H}$ into $\mathscr{H}$ with its metric topology). A subbase for the open sets consists of sets in the form

$$\{A\in\mathscr{B}(\mathscr{H}): \|(A-A_0)x\|<1, A_0\in\mathscr{B}(\mathscr{H}), x\in\mathscr{H}\}.$$

This is the strong-operator topology on $\mathscr{B}(\mathscr{H})$ (and convergence of $\{A_n\}$ to A in this topology amounts to convergence of $\{A_n x\}$ to Ax in $\mathscr{H}$ for each x in $\mathscr{H}$). A *von Neumann algebra* is a strong-operator-closed, self-adjoint subalgebra of $\mathscr{B}(\mathscr{H})$ containing I.

If $\{A_a: a\in\mathbb{A}\}$ is an increasing net of operators in a von Neumann algebra $\mathscr{R}$ (that is, $A_a\leq A_{a'}$ when $a\leq a'$) and there is some constant c such that $A_a\leq cI$ for each a in $\mathbb{A}$ (we say that $\{A_a\}$ is *bounded above* in this case), then $\{A_a\}$ converges to some operator A (necessarily in $\mathscr{R}$) in the strong-operator topology. The limit A of $\{A_a\}$ is also its least upper bound in the sense that $A_a\leq A$ for each a and if $A_a\leq A_0$ for each a, then $A\leq A_0$. Of course, the corresponding results hold for decreasing nets in $\mathscr{R}$.

The order property of von Neumann algebras just noted goes a long way toward characterizing them. A C^*-algebra $\mathfrak{A}$ with this property is generated by its projections – each self-adjoint operator in $\mathfrak{A}$ has a spectral resolution in $\mathfrak{A}$ – and it is tempting to guess that C^*-algebras with this property are isomorphic to von Neumann algebras. That guess is wrong as we shall see in the next section [4].

If the von Neumann algebra $\mathscr{R}$ acts on the Hilbert space $\mathscr{H}$, each unit vector x in $\mathscr{H}$ gives rise to a state ω_x of $\mathscr{R}$ defined by: $\omega_x(A)=\langle Ax, x\rangle$ $(A\in\mathscr{R})$. Such states have a special continuity property relative to the order structure on $\mathscr{R}$: $\{\omega_x(A_a)\}$ tends to $\omega_x(A)$ when $\{A_a\}$ is an increasing net in $\mathscr{R}$ with least upper bound A. More generally, if $\{x_n\}$ and $\{y_n\}$ are sequences of vectors in $\mathscr{H}$ such that $\sum\|x_n\|^2<\infty$ and $\sum\|y_n\|^2<\infty$, then the mapping $A\to\sum\langle Ax_n, y_n\rangle$ defines a linear functional ϱ on $\mathscr{R}$ with this same order continuity property ($\{\varrho(A_a)\}$ tends to $\varrho(A)$). The family $\mathscr{R}_*$ of functionals such as ϱ is a linear subspace of the (norm) dual space $\mathscr{R}^*$ of $\mathscr{R}$. With some effort, one can show that $\mathscr{R}_*$ is a norm-closed subspace of $\mathscr{R}^*$; thus $\mathscr{R}_\#$ is a Banach space [3]. The restriction to $\mathscr{R}_\#$ of the image of $\mathscr{R}$ under its natural injection in $\mathscr{R}^{**}$ yields a subspace of $(\mathscr{R}_\#)^*$. The composition of that injection and restriction is a linear isomorphism ψ of $\mathscr{R}$ into $(\mathscr{R}_\#)^*$. It results from the fact that

$$\|A\| = \sup\{|\langle Ax, y\rangle| : \|x\| = \|y\| = 1\}$$

that ψ is an isometry. A continuous linear functional on $\mathscr{R}_\#$ restricted to functionals of the form $A \to \langle Ax, y\rangle$ gives rise to a bounded conjugate-bilinear functional on $\mathscr{H}$, which corresponds to an operator (in $\mathscr{R}$) by virtue of the Riesz representation of such bilinear functionals. This leads to the conclusion that ψ maps $\mathscr{R}$ *onto* $(\mathscr{R}_\#)^*$.

From the preceding discussion, several properties of von Neumann algebras are apparent. Not only do they have the order properties discussed, but they have separating families of states that respect these order properties. In addition, each von Neumann algebra is the dual space of some Banach space. Each of these observations leads to a representation-free characterization of von Neumann algebras.

With ϱ a state of the C^*-algebra $\mathfrak{A}$, we write x_ϱ for the vector $I + \mathscr{L}$ in $\mathscr{H}_\varrho$ and note that $\omega_{x_\varrho} \circ \pi_\varrho = \varrho$. In addition, $\pi_\varrho(\mathfrak{A}) = \mathfrak{A}/\mathscr{L}$, which is dense in $\mathscr{H}_\varrho$.

Definition 2.1. A C^*-algebra $\mathscr{R}$ that satisfies the following two conditions is said to be a *W^*-algebra*:

(i) if $\{A_a\}$ is an increasing net of self-adjoint operators in $\mathscr{R}$ and $\{A_a\}$ has an upper bound, then $\{A_a\}$ has a least upper bound A in $\mathscr{R}$;

(ii) the set $\mathscr{N}$ of states ω of $\mathscr{R}$ with the property that $\{\omega(A_a)\}$ converges to $\omega(A)$ when $\{A_a\}$ is an increasing net in $\mathscr{R}$ with least upper bound A separates $\mathscr{R}$.

A state in $\mathscr{N}$ is said to be a *normal state* of $\mathscr{R}$.

The first characterization of von Neumann algebras [7] (Section 4) tells us that they are precisely the W^*-algebras. The second characterization [16] tells us that the von Neumann algebras are precisely those C^*-algebras that are dual spaces. In the next section, we construct a commutative C^*-algebra satisfying (i) of Definition 2.1 but not isomorphic to a von Neumann algebra.

3. The commutative case

A *boundedly complete lattice* is a lattice in which each non-empty family of elements that has an upper bound has a *least* upper bound. By studying the set of lower bounds, it follows that in a boundedly complete lattice, each non-empty family that has a lower bound has a *greatest* lower bound. We begin with a classical theorem of Stone's [20] concerning the algebra $C(X)$ of all complexvalued continuous functions on a compact Hausdorff space X (provided with the pointwise algebraic operations). When we refer to order properties of $C(X)$, the pointwise ordering is used and it is understood that the real subalgebra of real-valued functions in $C(X)$ is being considered.

Theorem 3.1. *If $C(X)$ is a boundedly complete lattice, then each open set in X has an open closure.*

Proof. Let $\mathcal{O}$ be an open subset of X, $\mathcal{O}^-$ its closure, $\mathcal{F}$ the family of functions f in $C(X)$ such that $0 \le f \le 1$ and $f(p') = 0$ if $p' \notin \mathcal{O}$, and f_0 the least upper bound of $\mathcal{F}$ in $C(X)$. Since 1 is an upper bound for $\mathcal{F}$, $f_0 \le 1$. If $p \in \mathcal{O}$ there is an f in $\mathcal{F}$ such that $f(p) = 1$, so that $f_0(p) = 1$ for each p in $\mathcal{O}$ (hence, for each p in $\mathcal{O}^-$). If $p' \notin \mathcal{O}^-$, there is a g in $C(X)$ such that $0 \le g \le 1$, $g(p') = 0$, and $g(p) = 1$ for p in $\mathcal{O}^-$. Thus g is an upper bound for $\mathcal{F}$, and $f_0 \le g$. It follows that f_0 is 1 on $\mathcal{O}^-$ and 0 on $X \setminus \mathcal{O}^-$. As f_0 is continuous, $\mathcal{O}^-$ is open.

When a space X has the property that each open set has an open closure, we say that it is *extremely disconnected.* A subset that is both closed and open is described as *clopen*. The theorem that follows establishes the converse to Theorem 3.1 [10, 20].

Theorem 3.2. *If X is an extremely disconnected compact Hausdorff space, then $C(X)$ is a boundedly complete lattice.*

Proof. Let $\{f_a : a \in \mathbb{A}\}$ be a family of real-valued functions in $C(X)$ bounded above by some constant. Our program (carried out in five steps) is to construct the "spectral resolution" of the function that should be the least upper bound of $\{f_a\}$.

(i) Suppose, first, that each $\{f_a\}$ is the characteristic function of some clopen subset X_a of X. We show that $[\bigcup_{a\in\mathbb{A}} X_a]^-$ is a clopen set whose characteristic function $\bigvee_{a\in\mathbb{A}} f_a$ is the least upper bound (in $C(X)$) of $\{f_a\}$ and that the interior of $\bigcap_{a\in\mathbb{A}} X_a$ is a clopen set whose characteristic function $\bigwedge_{a\in\mathbb{A}} f_a$ is the greatest lower bound of $\{f_a\}$ in $C(X)$. Since $\bigcup_{a\in\mathbb{A}} X_a$ is open, its closure Y_0 is clopen. If g is an upper bound for $\{f_a\}$, then $1 \le g(p)$ for each p in $\bigcup_{a\in\mathbb{A}} X_a$. By continuity of g, the same is true for each p in Y_0. Thus $\bigvee_{a\in\mathbb{A}} f_a$ is the least upper bound of $\{f_a\}$. Now $1 - f_a$ is the characteristic function of $X \setminus X_a$. Hence the least upper bound $1 - f_0$ of $\{1 - f_a\}$ is the characteristic function of $[\bigcup_{a\in\mathbb{A}} (X \setminus X_a)]^-$. Thus f_0, the characteristic function

$$X \setminus [\bigcup_{a\in\mathbb{A}} (X \setminus X_a)]^- \quad (= X \setminus [X \setminus \bigcap_{a\in\mathbb{A}} X_a]^-),$$

which is the interior of $\bigcap_{a\in\mathbb{A}} X_a$, is the greatest lower bound of $\{f_a\}$.

(ii) We show next that

$$X \setminus [\bigcup_{a\in\mathbb{A}} \{x \in X : f_a(x) > \lambda\}]^- \quad (= X_\lambda)$$

is a clopen subset of X and that if Y is a clopen subset of X with the property that $f_a(p) \le \lambda$ for all a in $\mathbb{A}$ and all p in Y, then $Y \subseteq X_\lambda$. Since X_λ is the complement in X of the closure of a union of open subsets of X, X_λ is clopen. If $p\in X_\lambda$, then for each a in $\mathbb{A}$, $p\notin\{x\in X: f_a(x)>\lambda\}$; that is, $f_a(p)\le\lambda$. By assumption on Y, $Y\subseteq X\backslash f_a^{-1}((\lambda,\infty))$ so that $f_a^{-1}((\lambda,\infty))\subseteq X\backslash Y$. As Y is open, X/Y is closed. Thus

$$[\bigcup_{a\in\mathbb{A}} f_a^{-1}((\lambda,\infty))]^-\subseteq X\backslash Y,$$

and $Y\subseteq X_\lambda$.

(iii) Let e_λ be the characteristic function of X_λ. Let k be a constant that bounds $\{f_a\}$ above and such that $-k\le f_{a'}$ for some a' in $\mathbb{A}$. We prove that

(1) $e_\lambda=0$ for $\lambda<-k$ and $e_\lambda=1$ for $\lambda>k$;

(2) $e_\lambda\le e_{\lambda'}$ if $\lambda\le\lambda'$;

(3) $e_\lambda=\bigwedge_{\lambda'>\lambda} e_{\lambda'}$.

If $\lambda<-k$ and $p\in X_\lambda$, then $p\notin\{x\in X: f_{a'}(x)>\lambda\}$. But $-k\le f_{a'}$, so $\{x\in X: f_{a'}(x)>\lambda\}=X$. Thus $X_\lambda=\emptyset$ and $e_\lambda=0$.

If $\lambda\ge k$ and $p\in X$, then $f_a(p)\le\lambda$ for all a in $\mathbb{A}$. Thus X is a clopen set on which all f_a take values not exceeding λ. From (ii), $X\subseteq X_\lambda$; so $X=X_\lambda$ and $e_\lambda=1$.

If $\lambda\le\lambda'$, then

$$\{x\in X: f_a(x)>\lambda'\}\subseteq\{x\in X: f_a(x)>\lambda\}$$

and $X_\lambda\subseteq X_{\lambda'}$. Hence $e_\lambda\le e_{\lambda'}$.

Since $e_\lambda\le e_{\lambda'}$ when $\lambda\le\lambda'$, $e_\lambda\le\bigwedge_{\lambda'>\lambda}e_{\lambda'}$. Thus $X_\lambda\subseteq Y_\lambda$, where Y_λ is the set whose characteristic function is $\bigwedge_{\lambda'>\lambda}e_{\lambda'}$. Now $Y_\lambda\subseteq X_{\lambda'}$ for each λ' greater than λ. Thus if $p\in Y_\lambda$, $f_a(p)\le\lambda'$ for each such λ' and each a in $\mathbb{A}$. Hence $f_a(p)\le\lambda$ for all a in $\mathbb{A}$, and Y_λ is a clopen set on which all f_a take values not exceeding λ. From (ii), $Y_\lambda\subseteq X_\lambda$. Hence $Y_\lambda=X_\lambda$, and $e_\lambda=\bigwedge_{\lambda'>\lambda}e_{\lambda'}$.

(iv) We show now that $\int_{-k}^{k}\lambda\,de_\lambda$ converges in norm (in the sense of approximating Riemann sums) to a function f in $C(X)$ and that X_λ is the largest clopen set on which f takes values not exceeding λ. If $\{\lambda_0,\dots,\lambda_n\}(=\mathscr{P})$ and $\{\mu_0,\dots,\mu_m\}(=\mathscr{Q})$ are partitions of $[-k,k]$, $|\mathscr{P}|$ and $|\mathscr{Q}|$ are the lengths of the largest subintervals, and $\{\gamma_0,\dots,\gamma_r\}$ is their common refinement, then

$$\left\|\sum_{j=1}^{n}\lambda_j'(e_{\lambda_j}-e_{\lambda_{j-1}})-\sum_{k=1}^{r}\gamma_k'(e_{\gamma_k}-e_{\gamma_{k-1}})\right\|\le|\mathscr{P}|$$

and

$$\left\|\sum_{j=1}^{m}\mu_j'(e_{\mu_j}-e_{\mu_{j-1}})-\sum_{k=1}^{r}\gamma_k'(e_{\gamma_k}-e_{\gamma_{k-1}})\right\|\le|\mathscr{Q}|$$

so that

$$\left\|\sum_{j=1}^{n}\lambda_j'(e_{\lambda_j}-e_{\lambda_{j-1}})-\sum_{k=1}^{m}\mu_k'(e_{\mu_k}-e_{\mu_{k-1}})\right\|\le|\mathscr{P}|+|\mathscr{Q}|.$$

Thus the family of approximating Riemann sums to $\int_{-k}^{k}\lambda\,de_\lambda$ indexed by their corresponding partition of $[-k,k]$ and the set of these partitions partially ordered (and directed) by refinement, forms a Cauchy net in the norm topology on $C(X)$. Since $C(X)$ is complete in its norm topology, this net converges in norm to a real-valued function f in $C(X)$. As each approximating Riemann sum has range in $[-k,k]$, the same is true of f and $f=\int_{-a}^{a}\lambda\,de_\lambda$ when $k\le a$.

Suppose now that $k\le a$ and $\lambda\in[-a,a]$. If $\{\lambda_0,\dots,\lambda_n\}$ is a partition of $[-a,a]$, with λ as some λ_k, such that $(g=)\sum_{j=1}^{n}\lambda'_j(e_{\lambda_j}-e_{\lambda_{j-1}})$ is close (in norm) to f; then $\|fe_\lambda-ge_\lambda\|$ is small and

$$ge_\lambda=\sum_{j=1}^{k}\lambda'_j(e_{\lambda_j}-e_{\lambda_{j-1}})\le\sum_{j=1}^{k}\lambda_k(e_{\lambda_j}-e_{\lambda_{j-1}})=\lambda(e_\lambda-e_{-a})=\lambda e_\lambda.$$

Thus $fe_\lambda\le\lambda e_\lambda$. At the same time, $\|f(1-e_\lambda)-g(1-e_\lambda)\|$ is small and

$$\begin{aligned}g(1-e_\lambda)&=\sum_{j=k+1}^{n}\lambda'_j(e_{\lambda_j}-e_{\lambda_{j-1}})\\&\ge\sum_{j=k+1}^{n}\lambda_k(e_{\lambda_j}-e_{\lambda_{j-1}})=\lambda(e_a-e_\lambda)=\lambda(1-e_\lambda).\end{aligned}$$

Thus $f(1-e_\lambda)\ge\lambda(1-e_\lambda)$.

Let Y_λ be $X\backslash f^{-1}((\lambda,\infty))^-$. Then Y_λ is a clopen subset of X on which f takes values not exceeding λ. If Y is another clopen subset of X on which f takes values not exceeding λ, then $Y\subseteq X\backslash f^{-1}((\lambda,\infty))$ so that $f^{-1}((\lambda,\infty))\subseteq X\backslash Y$. As Y is open $X\backslash Y$ is closed; and $f^{-1}((\lambda,\infty))^-\subseteq X\backslash Y$. Thus $Y\subseteq Y_\lambda$; and Y_λ is the *largest* clopen set in X on which f takes values not exceeding λ.

Since $fe_\lambda\le\lambda e_\lambda$, f takes values not exceeding λ on X_λ and $X_\lambda\subseteq Y_\lambda$. As $e_\lambda=\bigwedge_{\lambda'>\lambda}e_{\lambda'}$, X_λ is the largest clopen set in X contained in $\bigcap_{\lambda'>\lambda}X_{\lambda'}$. Now $\lambda'\le f(p)$ if $p\in X\backslash X_{\lambda'}$, since $\lambda'(1-e_{\lambda'})\le f(1-e_{\lambda'})$, so that $X\backslash X_{\lambda'}\subseteq f^{-1}((\lambda,\infty))^-$ when $\lambda'>\lambda$. Thus $Y_\lambda\subseteq X_{\lambda'}$ when $\lambda'>\lambda$; and Y_λ is a clopen set contained in $\bigcap_{\lambda'>\lambda}X_{\lambda'}$. Since X_λ is the largest such clopen set, $Y_\lambda\subseteq X_\lambda$. Hence $Y_\lambda=X_\lambda$, and X_λ is the largest clopen set on which f takes values not exceeding λ.

(v) We are now in a position to show that f is the least upper bound of $\{f_a\}$ and conclude that $C(X)$ is a boundedly complete lattice. If $f(p)<f_a(p)$ for some p in X and some a in $\mathbb{A}$, there is a λ and a clopen set Y containing p such that $f(q)\le\lambda$ and $\lambda<f_a(q)$ for each q in Y. From (iv), $p\in Y\subseteq X_\lambda$. But $p\in\{x\in X:f_a(x)>\lambda\}$, so $p\notin X_\lambda$, a contradiction. Thus $f_a\le f$ for each a in $\mathbb{A}$, and f is an upper bound for $\{f_a\}$.

If g is an upper bound for $\{f_a\}$ and $g(p)<f(p)$ for some p in X, then, again, there is a λ and a clopen set Y containing p such that $g(q)\le\lambda<f(q)$ for each q in Y. Since $f_a\le g$ for all a in $\mathbb{A}$, Y is a clopen set on which all f_a take values not exceeding λ. From (ii), $p\in Y\subseteq X_\lambda$. From (iv), $f(p)\le\lambda$, a contradiction. Thus $f\le g$.

and f is the least upper bound of $\{f_a\}$ in $C(X)$. It follows that $C(X)$ is a boundedly complete lattice.

The lemma that follows lists conditions on a compact Hausdorff space that are equivalent to its being extremely disconnected.

Lemma 3.3. *Let X be a compact Hausdorff space.*

(i) *X is extremely disconnected if and only if each pair of disjoint open sets have disjoint closures.*

(ii) *X is extremely disconnected if and only if it satisfies the following two conditions:*

(a) *X is is totally disconnected,*

(b) *the family $\mathscr{C}$ of clopen subsets of X partially ordered by inclusion is a complete lattice.*

Proof. (i) Suppose $\mathcal{O}_1$ and $\mathcal{O}_2$ are disjoint open subsets of X and X is extremely disconnected. Since $\mathcal{O}_2$ is open, $X\backslash\mathcal{O}_2$ is closed and contains $\mathcal{O}_1^-$. Thus $X\backslash\mathcal{O}_1^-$ contains $\mathcal{O}_2$. Since X is extremely disconnected, $\mathcal{O}_1^-$ is open so that $X\backslash\mathcal{O}_1^-$ is closed. Hence $X\backslash\mathcal{O}_1^-$ contains $\mathcal{O}_2^-$ — that is, $\mathcal{O}_1^-\cap\mathcal{O}_2^-=\emptyset$.

Suppose now that disjoint open subsets of X have disjoint closures and let $\mathcal{O}$ be an open subset of X. Then $\mathcal{O}$ and $X\backslash\mathcal{O}^-$ are disjoint open subsets of X. By assumption, $\mathcal{O}^-$ is disjoint from the closure F of $X\backslash\mathcal{O}^-$. But $F\cup\mathcal{O}^-=X$ (since $X=(X\backslash\mathcal{O}^-)\cup\mathcal{O}^-$). Hence $\mathcal{O}^-$ is the complement in X of F, and $\mathcal{O}^-$ is open. Thus X is extremely disconnected.

(ii) Assume that X is totally disconnected and $\mathscr{C}$ is a complete lattice. Since X is a compact Hausdorff space in which points can be separated by *clopen* subsets of X, a standard compactness argument (replacing "open" by "clopen") shows that a point can be separated from a closed (=compact) subset of X by clopen sets. In particular, each open set in X is a union of clopen sets. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be disjoint open subsets of X and let $\mathscr{C}_j$ be $\{X_0\in\mathscr{C}: X_0\subseteq\mathcal{O}_j\}$ for j in $\{1,2\}$. By assumption, $\mathscr{C}_1$ has a least upper bound X_1 in $\mathscr{C}$. If $X_0\in\mathscr{C}_2$, then $X\backslash X_0$ is a clopen set, contains $\mathcal{O}_1$, and, hence, contains each element of $\mathscr{C}_1$. Thus $X_1\subseteq X\backslash X_0$ and $X_0\subseteq X\backslash X_1$. Since $\mathcal{O}_2=\bigcup\mathscr{C}_2$, $\mathcal{O}_2\subseteq X\backslash X_1$. But $X\backslash X_1$ is clopen so that $\mathcal{O}_2^-\subseteq X\backslash X_1$. As $\mathcal{O}_1\subseteq X_1$ and X_1 is clopen, $\mathcal{O}_1^-\subseteq X_1$. Thus $\mathcal{O}_1^-\cap\mathcal{O}_2^-=\emptyset$. From (i), X is extremely disconnected.

Assume, now, that X is extremely disconnected. From Theorem 3.2, $\mathscr{C}$ is a complete lattice; and, of course X is totally disconnected.

In the results that follow, we give the details of the construction of a commutative C^*-algebra not isomorphic to a von Neumann algebra but isomorphic to $C(X)$ with X an extremely disconnected compact Hausdorff space. The formulation, organization, and arguments are our own; the basic ideas are not. An early version of this example can be found in [4]; but as far as this author can determine, the fundamental ingenuity of this class of examples is due to Birkhoff-Ulam as described in the pages leading to Corollary 1 in [1; p. 186].

In a topological space X, a subset is said to be *meager* when it is a subset of a countable union of subsets of X each of which is nowhere dense in X. An open subset of X is said to be *regular* when it coincides with the interior of its closure. (The interval $(0, 1)$ is a regular open set in $\mathbb{R}$, but $(-1, 0) \cup (0, 1)$ is not.)

Lemma 3.4. *Let X be a complete metric space.*

(i) *The interior of the closure of an open set and the interior of the complement of a regular open set in X are regular.*

(ii) *Each open subset of X differs from a regular open subset on a meager set.*

(iii) *Each Borel subset of X differs from a regular open subset on a meager (Borel) set.*

(iv) *There is a unique regular open subset of X that differs from a given Borel set on a meager (Borel) set.*

(v) *Let $\mathscr{F}_0$ be the family of regular open subsets of X partially ordered by inclusion. Then $\mathscr{F}_0$ is a complete lattice.*

(vi) *Let $\mathscr{F}$ be the family of Borel subsets of X and $\mathscr{M}$ the σ-ideal of meager Borel subsets of X (a countable union of sets in $\mathscr{M}$ is in $\mathscr{M}$ and the intersection of a set of $\mathscr{M}$ with any set of $\mathscr{F}$ is in $\mathscr{M}$). Let $\mathscr{F}/\mathscr{M}$ be the family of equivalence classes of sets in $\mathscr{F}$ under the relation: $S \sim S'$ when S and S' differ by a meager set. With $\mathscr{S}$ and $\mathscr{S}'$ in $\mathscr{F}/\mathscr{M}$, define: $\mathscr{S} \lesssim \mathscr{S}'$ when $S \subseteq S'$ for some S in $\mathscr{S}$ and S' in $\mathscr{S}'$. Then $\lesssim$ is a partial ordering of $\mathscr{F}/\mathscr{M}$ (the "quotient" of "inclusion" on $\mathscr{F}$ by the ideal $\mathscr{M}$), each $\mathscr{S}$ in $\mathscr{F}/\mathscr{M}$ contains precisely one regular open set, and the mapping that assigns to each $\mathscr{S}$ in $\mathscr{F}/\mathscr{M}$ the regular open set it contains is an order isomorphism of $\mathscr{F}/\mathscr{M}$ onto $\mathscr{F}_0$. The partially ordered set $\mathscr{F}/\mathscr{M}$ is a complete lattice.*

(vii) *The algebra $\mathscr{B}(X)$ of bounded Borel functions on X is a commutative C^*-algebra and the family $\mathscr{M}_0$ of functions in $\mathscr{B}(X)$ that vanish on the complement of a meager Borel set is a closed ideal in $\mathscr{B}(X)$ and $\mathscr{B}(X)/\mathscr{M}_0$ is a commutative C^*-algebra.*

(viii) *Let Y be the compact Hausdorff space such that $\mathscr{B}(X)/\mathscr{M}_0 \cong C(Y)$. Then Y is totally disconnected and the family of clopen subsets of Y, partially ordered by inclusion, is a complete lattice, Y is extremely disconnected, and $C(Y)$ (and $\mathscr{B}(X)/\mathscr{M}_0$) are boundedly complete lattices.*

Proof. (i) Let Y be a closed subset of X and $\mathcal{O}$ be its interior. Since $\mathcal{O}$ is an open subset of X contained in $\mathcal{O}^-$, $\mathcal{O}$ is contained in the interior $\mathcal{O}_0$ of $\mathcal{O}^-$. Since $\mathcal{O}_0 \subseteq \mathcal{O}^- \subseteq Y$ and $\mathcal{O}_0$ is an open subset of X, $\mathcal{O}_0$ is contained in the interior $\mathcal{O}$ of Y. Thus $\mathcal{O} = \mathcal{O}_0$, $\mathcal{O}$ is the interior of $\mathcal{O}^-$, and $\mathcal{O}$ is regular. Both assertions of (i) follow.

(ii) Let $\mathcal{O}$ be an open subset of X and $\mathcal{O}_0$ be the interior of $\mathcal{O}^-$. Then

$$(\mathcal{O}_0\backslash\mathcal{O})\cup(\mathcal{O}\backslash\mathcal{O}_0)=\mathcal{O}_0\backslash\mathcal{O}\subseteq\mathcal{O}^-\backslash\mathcal{O}.$$

But $\mathcal{O}^-\backslash\mathcal{O}$ is a (closed) nowhere-dense set. Hence $\mathcal{O}$ and $\mathcal{O}_0$ differ on a meager subset of X (that is, $\mathcal{O}\sim\mathcal{O}_0$). From (i), $\mathcal{O}_0$ is regular.

(iii) Let $\mathscr{F}'$ be the family of Borel subsets of X that differ from a regular open set by a meager (Borel) set. If $S\in\mathscr{F}'$ and $\mathcal{O}_0$ is a regular open set such that $S\sim\mathcal{O}_0$, then $(S\backslash\mathcal{O}_0)\cup(\mathcal{O}_0\backslash S)$ $(=[(X\backslash S)\backslash(X\backslash\mathcal{O}_0)]\cup[(X\backslash\mathcal{O}_0)\backslash(X\backslash S)])$ is meager. Thus $X\backslash S\sim X\backslash\mathcal{O}_0$. From (i), the interior $\mathcal{O}_1$ of $X\backslash\mathcal{O}_0$ is regular, and $\mathcal{O}_1\sim X\backslash\mathcal{O}_0\sim X\backslash S$. Thus $X\backslash S\in\mathscr{F}'$.

Suppose $S_1,S_2,\ldots$ are in $\mathscr{F}'$. Let $\mathcal{O}_j$ be a regular open set such that $S_j\sim\mathcal{O}_j$. Then $(S_j\backslash\mathcal{O}_j)\cup(\mathcal{O}_j\backslash S_j)$ $(=M_j)$ is meager and

$$\left[\left(\bigcup_{j=1}^{\infty}S_j\right)\Big\backslash\left(\bigcup_{j=1}^{\infty}\mathcal{O}_j\right)\right]\cup\left[\left(\bigcup_{j=1}^{\infty}\mathcal{O}_j\right)\Big\backslash\left(\bigcup_{j=1}^{\infty}S_j\right)\right]\subseteq\bigcup_{j=1}^{\infty}M_j.$$

As $\bigcup_{j=1}^{\infty}M_j$ is meager, we have $\bigcup_{j=1}^{\infty}S_j\sim\bigcup_{j=1}^{\infty}\mathcal{O}_j$. From (i), the interior $\mathcal{O}_0$ of $(\bigcup_{j=1}^{\infty}\mathcal{O}_j)^-$ is regular and $\mathcal{O}_0\sim(\bigcup_{j=1}^{\infty}\mathcal{O}_j)^-\sim\bigcup_{j=1}^{\infty}S_j$. Thus $\bigcup_{j=1}^{\infty}S_j\in\mathscr{F}'$ and $\mathscr{F}'$ is a σ-algebra containing the open sets and contained in $\mathscr{F}$. It follows that $\mathscr{F}'=\mathscr{F}$.

(iv) If $S\in\mathscr{F}, S\sim\mathcal{O}_1$, $S\sim\mathcal{O}_2$, and $\mathcal{O}_1$ and $\mathcal{O}_2$ are regular open sets, then $\mathcal{O}_1\sim\mathcal{O}_2$. Since $\mathcal{O}_2^-$ is closed, if some p in $\mathcal{O}_1$ is not in $\mathcal{O}_2^-$, then some open set $\mathcal{O}$ containing p does not meet $\mathcal{O}_2^-$, and $\mathcal{O}\cap\mathcal{O}_1\subseteq\mathcal{O}_1\backslash\mathcal{O}_2$. But $\mathcal{O}_1\backslash\mathcal{O}_2$ is meager, and meager sets in a complete metric space have null interior. Thus $\mathcal{O}_1\subseteq\mathcal{O}_2^-$ and $\mathcal{O}_1$ is contained in the interior $\mathcal{O}_2$ of $\mathcal{O}_2^-$. Symmetrically, $\mathcal{O}_2\subseteq\mathcal{O}_1$ and $\mathcal{O}_1=\mathcal{O}_2$.

(v) Suppose $\mathcal{O}_a\in\mathscr{F}_0$ for a in $\mathbb{A}$. Let $\mathcal{O}_1$ be the interior of $(\bigcup_{a\in\mathbb{A}}\mathcal{O}_a)^-$. Then $\mathcal{O}_1\in\mathscr{F}_0$, from (i), since $\bigcup_{a\in\mathbb{A}}\mathcal{O}_a$ is open. Clearly $\mathcal{O}_1$ is an upper bound for $\{\mathcal{O}_a: a\in\mathbb{A}\}$. If $\mathcal{O}$ is another upper bound, then $(\bigcup_{a\in\mathbb{A}}\mathcal{O}_a)^-\subseteq\mathcal{O}^-$ and the interior $\mathcal{O}$ of $\mathcal{O}^-$ contains the interior $\mathcal{O}_1$ of $(\bigcup_{a\in\mathbb{A}}\mathcal{O})^-$. Thus $\mathcal{O}_1$ is the least upper bound of $\{\mathcal{O}_a: a\in\mathbb{A}\}$ and $\mathscr{F}_0$ is a complete lattice. (Note that the set of all lower bounds of $\{\mathcal{O}_a\}$ has a least upper bound $\mathcal{O}_0$ and $\mathcal{O}_0$ is the greatest lower bound of $\{\mathcal{O}_a\}$.)

(vi) As defined, the relation $\lesssim$ is clearly reflexive and transitive. Suppose $\mathscr{S}\lesssim\mathscr{S}'$ and $\mathscr{S}'\lesssim\mathscr{S}$, with $\mathscr{S}$ and $\mathscr{S}'$ in $\mathscr{F}/\mathscr{M}$. Then there are S_1 and S_2 in $\mathscr{S}$ and S_1' and S_2' in $\mathscr{S}'$ such that $S_1\subseteq S_1'$ and $S_2'\subseteq S_2$. Let M be $(S_1\backslash S_2)\cup(S_2\backslash S_1)$ and M' be $(S_1'\backslash S_2')\cup(S_2'\backslash S_1')$. Since $S_1\sim S_2$ and $S_1'\sim S_2'$; M, M', and $M\cup M'$ are meager, and $S_1\cup M=S_2\cup M$, $S_1'\cup M'=S_2'\cup M'$. Thus

$$S_1'\cup M'\cup M=S_2'\cup M'\cup M\subseteq S_2\cup M'\cup M=S_1\cup M'\cup M\subseteq S_1'\cup M'\cup M,$$

so that $S_1\sim S_1'$ and $\mathscr{S}=\mathscr{S}'$. Hence $\lesssim$ is a partial ordering of $\mathscr{F}/\mathscr{M}$.

From (iii) and (iv), each $\mathscr{S}$ in $\mathscr{F}$ differs from a unique regular open set $\mathcal{O}$ by a meager set. Thus the equivalence class $\mathscr{S}$ of S contains $\mathcal{O}$ and no other regular open set. If $\mathscr{S}'$ is another equivalence class and $\mathcal{O}'$ is the regular open set it contains, then $\mathscr{S}\precsim\mathscr{S}'$ if $\mathcal{O}\subseteq\mathcal{O}'$, by definition of $\precsim$. Conversely, if $\mathscr{S}\precsim\mathscr{S}'$, then $\mathcal{O}\cup M\subseteq\mathcal{O}'\cup M'$, where M and M' are in $\mathscr{M}$. Thus $\mathcal{O}\subseteq\mathcal{O}'\cup M'$ so that $\mathcal{O}\backslash\mathcal{O}'^{-}\subseteq\mathcal{O}\backslash\mathcal{O}'\subseteq M'$. Since $\mathcal{O}\backslash\mathcal{O}'^{-}$ is open and M' is meager, $\mathcal{O}\backslash\mathcal{O}'^{-}=\emptyset$, that is $\mathcal{O}\subseteq\mathcal{O}'^{-}$. Hence $\mathcal{O}$ is contained in the interior $\mathcal{O}'$ of $\mathcal{O}'^{-}$. It follows that the mapping $\mathscr{S}\to\mathcal{O}$ of $\mathscr{F}/\mathscr{M}$ onto $\mathscr{F}_0$ is an order isomorphism and, from (v), $\mathscr{F}/\mathscr{M}$ is a complete lattice.

(vii) If $\mathscr{B}(X)$ is provided with the supremum norm it becomes a Banach algebra. The operation of complex conjugation of functions is an adjoint operation on $\mathscr{B}(X)$. Since $\|f\bar{f}\|=\||f|^2\|=\||f|\|^2=\|f\|^2$, $\mathscr{B}(X)$ with the given norm and adjoint operation is a C^*-algebra. If f_1 and f_2 in $\mathscr{M}_0$ vanish outside of the meager sets M_1 and M_2, respectively, then f_1+f_2 vanishes outside $M_1\cup M_2$, a meager subset of X, and ff_1 vanishes outside M_1 for each f in $\mathscr{B}(X)$. Thus $\mathscr{M}_0$ is an ideal in $\mathscr{B}(X)$. If $f_n\in\mathscr{M}_0$ and $\|f-f_n\|\to 0$, then f vanishes outside $\bigcup_{n=1}^{\infty}M_n$, a meager set, where M_n is a meager set on the complement of which f_n vanishes. Thus $f\in\mathscr{M}_0$ and $\mathscr{M}_0$ is a closed (two-sided) ideal in $\mathscr{B}(X)$. With its (supremum) norm and complex-conjugation as involution, $\mathscr{B}(X)$ satisfies the conditions of the Gelfand-Neumark theorem [5]. Thus $\mathscr{B}(X)$ and hence $\mathscr{B}(X)/\mathscr{M}_0$ are commutative C^*-algebras.

(viii) Let $\mathscr{S}$ be in $\mathscr{F}/\mathscr{M}$ and e be the characteristic function of a set in $\mathscr{S}$. Define $\eta(\mathscr{S})$ to be the projection in $\mathscr{B}(X)/\mathscr{M}_0$ that is the image of e under the quotient mapping. If e' corresponds to another set in $\mathscr{S}$, then $e-e'\in\mathscr{M}_0$ so that e and e' have the same image in $\mathscr{B}(X)/M_0$ and $\eta(\mathscr{S})$ is well defined. If $\mathscr{S}\precsim\mathscr{S}'_{\eta}$, there are sets S in $\mathscr{S}$ and S' in $\mathscr{S}'$ such that $S\subseteq S'$. With e and e' the characteristic functions of S and S', respectively $e\leq e'$ so that $\eta(\mathscr{S})\leq\eta(\mathscr{S}')$.

Let E be a projection in $\mathscr{B}(X)/\mathscr{M}_0$ and f be an element of $\mathscr{B}(X)$ mapping onto E. Then f^2-f maps onto 0 and f^2-f vanishes outside some meager Borel set M. Let $e(p)$ be $f(p)$ for p in $X\backslash M$ and 0 for p in M. Then e is an idempotent in $\mathscr{B}(X)$ so that e is the characteristic function of a set S in $\mathscr{F}$. If $\mathscr{S}$ in $\mathscr{F}/\mathscr{M}$ is the equivalence class of S, then $\eta(\mathscr{S})=E$. Hence η is an order-preserving mapping of $\mathscr{F}/\mathscr{M}$ *onto* the set $\mathscr{P}$ of projections in $\mathscr{B}(X)/\mathscr{M}_0$.

If E and E' are in $\mathscr{P}$ and $E\leq E'$, there are $\mathscr{S}$ and $\mathscr{S}'$ in $\mathscr{F}/\mathscr{M}$ such that $\eta(\mathscr{S})=E$ and $\eta(\mathscr{S}')=E'$. By definition of η, there are sets S and S' in $\mathscr{S}$ and $\mathscr{S}'$ whose characteristic functions e and e' map onto E and E', respectively. Thus $2(e-ee')$ maps onto $(E'-E)^2-(E'-E)$ $(=0)$ and $e-ee'$ is 0 on $X\backslash M'$ for some meager set M'. It follows that $S\backslash S'\subseteq M'$ so that $S\subseteq S'\cup M'$. Since $S'\cup M'\in\mathscr{S}'$, $\mathscr{S}\precsim\mathscr{S}'$. Hence η is one-to-one, for if $\eta(\mathscr{S})=\eta(\mathscr{S}')=E$, then $\mathscr{S}\precsim\mathscr{S}'$ and $\mathscr{S}'\precsim\mathscr{S}$ so that $\mathscr{S}=\mathscr{S}'$ from (vi). It follows, too, that η^{-1} is order preserving.

If ϕ is the isomorphism of $\mathscr{B}(X)/\mathscr{M}_0$ onto $C(Y)$, then $\phi\circ\eta$ is an order isomorphism of $\mathscr{F}/\mathscr{M}$ with the set of idempotents $\mathscr{P}'$ in $C(Y)$. From (vi), $\mathscr{P}'$ is a complete lattice. Each function in $\mathscr{B}(X)$ is approximable in norm as closely as we wish by step functions. Thus linear combinations of idempotents lie dense in $\mathscr{B}(X)$, $\mathscr{B}(X)/\mathscr{M}_0$, and in $C(Y)$. Hence Y is totally disconnected. From Lemma 3.3

(ii), Y is extremely disconnected. Thus $C(Y)$ and $\mathscr{B}(X)/\mathscr{M}_0$ are boundedly complete lattices, by Theorem 3.2.

Theorem 3.5. *With the notation of Lemma 3.4, assume that X is $[0,1]$ and let ϱ be a state of $C(Y)$.*

(i) *Suppose $\varrho(\bigvee_{n=1}^{\infty} e_n)=\sum_{n=1}^{\infty}\varrho(e_n)$ whenever $\{e_n\}$ is a countable family of idempotents in $C(Y)$ such that $e_n \cdot e_{n'}=0$ unless $n=n'$. (We shall say that ϱ is a normal state in this case.) Then $\varrho(\bigvee_{n=1}^{\infty} f_n)\le\sum_{n=1}^{\infty}\varrho(f_n)$ for each countable set $\{f_n\}$ of idempotents f_n in $C(Y)$ (where '$a\le+\infty$' is envisaged in the inequality of this assertion).*

(ii) *Enumerate the open intervals in $[0,1]$ $(=X)$ with rational endpoints and let $f_1, f_2, \dots$ be the idempotents in $C(Y)$ that are the images of their characteristic functions (in $\mathscr{B}(X)$) under the composition of the quotient mapping of $\mathscr{B}(X)$ onto $\mathscr{B}(X)/\mathscr{M}_0$ and the isomorphism of $\mathscr{B}(X)/\mathscr{M}_0$ with $C(Y)$. For each j in $\{1,2,\dots\}$, let e_j be an idempotent in $C(Y)$ such that $0<e_j\le f_j$. Then $\bigvee_{j=1}^{\infty} e_j=1$.*

(iii) *With the notation of* (ii) *and given a positive ε, e_j can be chosen such that $\varrho(e_j)\le 2^{-j}\varepsilon$ and $C(Y)$ has no normal states.*

(iv) *The C^*-algebra $C(Y)$ is isomorphic to no abelian von Neumann algebra although Y is extremely disconnected.*

Proof. (i) Let f_1' be f_1 and f_n' be $f_1\vee\dots\vee f_n - f_1\vee\dots\vee f_{n-1}$ for n in $\{2,3,\dots\}$. If $m<n$, then $f_m'\le f_1\vee\dots\vee f_{n-1}$ so that $f_m'\cdot f_n'=0$. Moreover, $f_1'+\dots+f_n'=f_1\vee\dots\vee f_n$ for each n in $\{1,2,\dots\}$ so that $\bigvee_{n=1}^{\infty} f_n=\bigvee_{n=1}^{\infty} f_n'$ and

$$\varrho\left(\bigvee_{n=1}^{\infty} f_n\right)=\varrho\left(\bigvee_{n=1}^{\infty} f_n'\right)=\sum_{n=1}^{\infty}\varrho(f_n').$$

Now $f_1\vee\dots\vee f_n\le f_1+\dots+f_n$ so that

$$\sum_{n=1}^{m}\varrho(f_n')=\varrho(f_1\vee\dots\vee f_m)\le\sum_{n=1}^{m}\varrho(f_n)\le\sum_{n=1}^{\infty}\varrho(f_n).$$

Hence

$$\varrho\left(\bigvee_{n=1}^{\infty} f_n\right)=\sum_{n=1}^{\infty}\varrho(f_n')\le\sum_{n=1}^{\infty}\varrho(f_n).$$

(ii) With the notation (and results) of the proof of Lemma 3.4 (viii), let $\mathscr{S}_j$ be $(\phi\circ\eta)^{-1}(e_j)$. From Lemma 3.4 (iii), $\mathscr{S}_j$ contains a regular open set $\mathscr{O}_j$. Let $\mathscr{O}$ be $\bigcup_{j=1}^{\infty}\mathscr{O}_j$. If $p\in[0,1]\setminus\mathscr{O}^-$, then some open interval (a,b) with rational endpoints contains p and does not meet $\mathscr{O}$. Let $\mathscr{S}$ be the equivalence class of (a,b) in $\mathscr{F}/\mathscr{M}$ and f_j be $(\phi\circ\eta)(\mathscr{S})$. Since $0<e_j\le f_j$, $\mathscr{S}_j\lesssim\mathscr{S}$. Now (a,b) is regular and from Lemma 3.4 (iv), (a,b) is the only regular set in $\mathscr{S}$. From Lemma 3.4 (vi), $\mathscr{O}_j\subseteq(a,b)$ contradicting the choice of (a,b) (not meeting $\mathscr{O}$). Thus $\mathscr{O}^-=[0,1]$, $\bigvee_{j=1}^{\infty}\mathscr{O}_j=[0,1]$, and $\bigvee_{j=1}^{\infty} e_j=1$.

(iii) We note, first, that no non-zero idempotent f in $C(Y)$ is minimal. If f were minimal, $(\phi \circ \eta)^{-1}(f)$ $(=\mathscr{S})$ would be minimal in $\mathscr{F}/\mathscr{M}$ and the regular open set $\mathcal{O}$ in $\mathscr{S}$ would be non-empty and minimal in $\mathscr{F}_0$. But $\mathcal{O}$ contains some open interval (a,b) as a proper subset and as noted in (ii), (a,b) is regular. Thus we can choose an idempotent f' in $C(Y)$ such that $0<f'<f$. One of $\varrho(f')$ and $\varrho(f-f')$ is not greater than $\frac{1}{2}\varrho(f)$. Continuing this "division" process, we find an idempotent f'' in $C(Y)$ such that $0<f''<f$ and $\varrho(f'')<\varepsilon$. Applying this to f_j, we find e_j as described.

(iv) From (i), and with with the notation of (iii),

$$1=\varrho(1)=\varrho\left(\bigvee_{j=1}^{\infty} e_j\right)\leq \sum_{j=1}^{\infty} \varrho(e_j)\leq \sum_{j=1}^{\infty} 2^{-j}\varepsilon=\varepsilon,$$

a contradiction. Thus $C(Y)$ has no normal states. Since vector states of an abelian von Neumann algebra are normal, $C(Y)$ is isomorphic to no such algebra.

4. Characterizations of von Neumann algebras

We begin with a proof of the characterization found in [7]. Our first goal, reached in Lemma 4.3, is to show that a "monotone closed" C^*-algebra acting on a Hilbert space is a von Neumann algebra. With that lemma established, it is not difficult to use representation techniques to complete the characterization proof (Theorem 4.11).

The monotone closure methods of the next lemmas were later sharpened brilliantly by G.K. Pedersen to yield his stronger version [13] of Lemma 4.3: to pass from a represented C^*-algebra to its strong-operator closure it suffices to adjoin the limits of increasing nets then of decreasing nets and then of increasing nets. In the separable case a "sweep up and then down" (and with sequences) suffices. In a later article [14], Pedersen proves (by allied techniques) the much sought after result that each represented C^*-algebra whose maximal abelian subalgebras are strong-operator closed is itself strong-operator closed. In effect, Lemma 4.3 says that to check whether or not a represented C^*-algebra is strong-operator closed it suffices to check limits of monotone nets; Pedersen's result [14] says that it suffices to do this with monotone nets of commuting elements, while [13] supplies a finite-stage procedure for passing to the closure.

Lemma 4.1. *Let $\mathfrak{A}$ be a C^*-algebra acting on a Hilbert space $\mathscr{H}$. Suppose that each increasing net of operators in $\mathfrak{A}$ that is bounded above has its strong-operator limit in $\mathfrak{A}$.*

Then:

(i) *each decreasing net of operators in $\mathfrak{A}$ that is bounded below has its strong-operator limit in $\mathfrak{A}$;*

(ii) *the range projection of each operator in* $\mathfrak{A}$ *lies in* $\mathfrak{A}$;

(iii) *the union and intersection of each finite set of projections in* $\mathfrak{A}$ *lie in* $\mathfrak{A}$;

(iv) *the union and intersection of an arbitrary set of projections in* $\mathfrak{A}$ *lie in* $\mathfrak{A}$;

(v) $E \in \mathfrak{A}$, *where* E *is a cyclic projection in* $\mathfrak{A}^-$ *with generating vector* x, *provided that for each vector* y *in* $(I-E)(\mathscr{H})$ *there is a self-adjoint* A_y *in* $\mathfrak{A}$ *such that* $A_y x = x$ *and* $A_y y = 0$;

(vi) $\mathfrak{A}^- = \mathfrak{A}$ *if each cyclic projection in* $\mathfrak{A}^-$ *lies in* $\mathfrak{A}$.

Proof. (i) if $\{A_a\}$ is a decreasing net in $\mathfrak{A}$ that is bounded below with strong-operator limit A, then $\{-A_a\}$ is an increasing net in $\mathfrak{A}$ with strong-operator limit $-A$. By assumption, $-A \in \mathfrak{A}$, so that $A \in \mathfrak{A}$.

(ii) From [8; Proposition 2.5.13], $R(A^*) = R(A^*A)$, so that it suffices to show that $R(H) \in \mathfrak{A}$ for each positive H in $\mathfrak{A}$. Of course $R(H) = R(aH)$ for each positive scalar a. Thus we may assume that $0 \leq H \leq I$. In this case $\{H^{1/n}\}$ is a monotone increasing sequence in $\mathfrak{A}$ and from [8; Lemma 5.1.5], $R(H)$ is its strong-operator limit. By assumption then, $R(H) \in \mathfrak{A}$.

(iii) From [8; Proposition 2.5.14], $E \vee F = R(E+F) \in \mathfrak{A}$, when E and F are projections in $\mathfrak{A}$. Thus the union of a finite family of projections in $\mathfrak{A}$ is in $\mathfrak{A}$. Since $I - \bigvee_a (I - E_a) = \bigwedge_a E_a$, the intersection of a finite family of projections in $\mathfrak{A}$ is in $\mathfrak{A}$.

(iv) If $\{E_a : a \in \mathbb{A}\}$ is a collection of projections in $\mathfrak{A}$, the union of each finite subcollection lies in $\mathfrak{A}$ from (iii). The family of such unions, indexed by the family of finite subsets of $\mathbb{A}$ directed by inclusion is an increasing net with strong-operator limit $\bigvee_{a \in \mathbb{A}} E_a$. By assumption then, $\bigvee_{a \in \mathbb{A}} E_a \in \mathfrak{A}$.

(v) From our assumption that $A_y x = x$, $R(A_y)x = x$. Since $A_y y = 0$ and $A_y = A_y^*$, $R(A_y)y = 0$. Thus $Gx = x$ and $Gy = 0$ for each y in $(I-E)(\mathscr{H})$, where $G = \bigwedge_{y \in (I-E)(\mathscr{H})} R(A_y)$. From (ii) and (iv), $G \in \mathfrak{A}$. As E is cyclic under $\mathfrak{A}'$ with generating vector x, $E \leq G$. As $Gy = 0$ for each y in $(I-E)(\mathscr{H})$, $G \leq E$. Thus $E = G \in \mathfrak{A}$.

(vi) From [8; Proposition 5.5.9] and (iv), each projection in $\mathfrak{A}^-$ lies in $\mathfrak{A}$. From [8; Theorem 5.2.2 (v)], each self-adjoint operator in $\mathfrak{A}^-$ lies in $\mathfrak{A}$. Since $\mathfrak{A}^-$ is a self-adjoint algebra containing $\mathfrak{A}$, $\mathfrak{A}^- = \mathfrak{A}$.

Lemma 4.2. *Let* $\mathfrak{A}$ *and* $\mathscr{H}$ *be as in Lemma* 4.1. *Suppose* E *is a cyclic projection in* $\mathfrak{A}^-$ *and* x *is a unit generating vector for* $E(\mathscr{H})$ *under* $\mathfrak{A}'$. *With* y *a unit vector in* $(I-E)(\mathscr{H})$, *we have:*

(i) *there is a sequence* $\{A_n\}$ *in* $(\mathfrak{A}_h)_1$ *such that* $A_n x \to x$, $A_n y \to 0$,

$$\|(A_n - A_{n-1})^+ x\| < 2^{1-n}, \quad \textit{and} \quad \|(A_n - A_{n-1})^+ y\| < 2^{1-n},$$

where $A_0 = 0$;

(ii) $\{T_n\}$ *is a bounded monotone decreasing sequence of positive elements of* $\mathfrak{A}$, *where*

$$T_n = \left(I + \sum_{k=1}^{n} (A_k - A_{k-1})^+\right)^{-1}, \quad \textit{and} \quad T^{1/2}\left(\sum_{k=1}^{n} (A_k - A_{k-1})^+\right) T^{1/2} \le I$$

for each n, where T is the strong-operator limit of $\{T_n\}$ (*in* $\mathfrak{A}$);

(iii) *for each j in* $\{1, \ldots, n\}$, $\{T^{1/2}(\sum_{k=j}^{n}(A_k - A_{k-1})^+)T^{1/2}\}$ *is monotone increasing with n, bounded above by I, and if* C_j *is its strong-operator limit, then* $0 \le C_j \le I$, $\{C_j\}$ *is decreasing,*

$$T^{1/2}\left(\sum_{k=1}^{n} (A_k - A_{k-1})^+\right) T^{1/2} + C_{n+1} = C_1$$

and

$$T^{1/2} A_n T^{1/2} + C_{n+1} = T^{1/2}\left(\sum_{k=1}^{n} -(A_k - A_{k-1})^-\right) T^{1/2} + C_1;$$

(iv) $\{T^{1/2} A_n T^{1/2} + C_{n+1}\}$ *is monotone decreasing and bounded and* $T^{1/2} A T^{1/2} \in \mathfrak{A}$, *where A is a weak-operator limit point of* $\{A_n\}$;

(v) $R(T) \in \mathfrak{A}$, $R(T)x = x$, $R(T)y = y$;

(vi) *each maximal abelian (self-adjoint) subalgebra of* $\mathfrak{A}$ *is weak-operator closed.*

Proof. (i) From the Kaplansky density theorem, there is a sequence $\{A_n\}$ in $(\mathfrak{A}_h)_1$ such that $A_n x \to x$ and $A_n y \to 0$ since $E \in (\mathfrak{A}_h^-)_1$, $Ex = x$, and $Ey = 0$. Passing to a subsequence of $\{A_n\}$ (using the Cauchy criterion on the convergent subsequences, $\{A_n x\}$ and $\{A_n y\}$), we can arrange that

$$\|(A_n - A_{n-1})x\| < 2^{1-n}, \qquad \|(A_n - A_{n-1})y\| < 2^{1-n}.$$

For each self-adjoint A, A^+ and A^- have orthogonal ranges so that

$$\|Az\|^2 = \|A^+ z\|^2 + \|A^- z\|^2.$$

Thus $\|A^+ z\| \le \|Az\|$. It follows that for each n in $\mathbb{N}$,

$$\|(A_n - A_{n-1})^+ x\| < 2^{1-n}, \qquad \|(A_n - A_{n-1})^+ y\| < 2^{1-n}.$$

(ii) Since

$$I \le I + \sum_{k=1}^{n} (A_k - A_{k-1})^+ \le I + \sum_{k=1}^{n+1} (A_k - A_{k-1})^+,$$

we have from [8; Proposition 4.2.8 (iii)] that $0 \le T_{n+1} \le T_n \le I$. With u a given unit vector in $\mathscr{H}$ and m large enough $T_m^{1/2} u$ is close to $T^{1/2} u$ since the mapping $A \to A^{1/2}$ is strong-operator continuous on the unit ball of $\mathscr{B}(\mathscr{H})^+$ from [8: Proposition 5.3.2]. (In fact, this mapping is strong-operator continuous on $\mathscr{B}(\mathscr{H})^+$.) Now when $n \le m$,

$$\left\langle T_m^{1/2}\left(\sum_{k=1}^{n} (A_k - A_{k-1})^+\right) T_m^{1/2} u, u\right\rangle \le \left\langle T_m^{1/2}\left(\sum_{k=1}^{m} (A_k - A_{k-1})^+\right) T_m^{1/2} u, u\right\rangle$$

$$= \left\langle T_m\left(\sum_{k=1}^{m} (A_k - A_{k-1})^+\right) u, u\right\rangle \le \langle Iu, u\rangle;$$

and as $m\to\infty$,

$$\left\langle T_m^{1/2}\left(\sum_{k=1}^{n}(A_k-A_{k-1})^+\right)T_m^{1/2}u,u\right\rangle\to\left\langle T^{1/2}\left(\sum_{k=1}^{n}(A_k-A_{k-1})^+\right)T^{1/2}u,u\right\rangle.$$

Thus $T^{1/2}(\sum_{k=1}^{n}(A_k-A_{k-1})^+)T^{1/2}\le I$ for each n in $\mathbb{N}$.

(iii) From (ii), for each j in $\{1,\dots,n\}$

$$T^{1/2}\left(\sum_{k=j}^{n}(A_k-A_{k-1})^+\right)T^{1/2}\le T^{1/2}\left(\sum_{k=1}^{n}(A_k-A_{k-1})^+\right)T^{1/2}\le I.$$

Thus $\{T^{1/2}(\sum_{k=j}^{n}(A_k-A_{k-1})^+)T^{1/2}\}$ is an increasing sequence (over n) of operators in $\mathfrak{A}$ bounded above by I. Its limit C_j lies in $\mathfrak{A}$ and $0\le C_j\le I$. Since

$$\sum_{k=j+1}^{n}(A_k-A_{k-1})^+\le\sum_{k=j}^{n}(A_k-A_{k-1})^+$$

for each j and all n, $\{C_j\}$ is decreasing. In addition, for each n and m,

$$T^{1/2}\left(\sum_{k=1}^{n}(A_k-A_{k-1})^+\right)T^{1/2}+T^{1/2}\left(\sum_{j=n+1}^{m}(A_k-A_{k-1})^+\right)T^{1/2}$$
$$=T^{1/2}\left(\sum_{k=1}^{m}(A_k-A_{k-1})^+\right)T^{1/2}.$$

Thus $T^{1/2}(\sum_{k=1}^{n}(A_k-A_{k-1})^+)T^{1/2}+C_{n+1}=C_1$. Now $A_n-A_{n-1}=(A_n-A_{n-1})^+-(A_n-A_{n-1})^-$ and $A_0=0$, hence

$$T^{1/2}A_nT^{1/2}+C_{n+1}=T^{1/2}\left(\sum_{k=1}^{n}(A_k-A_{k-1})\right)T^{1/2}+C_{n+1}$$
$$=T^{1/2}\left(\sum_{k=1}^{n}-(A_k-A_{k-1})^-\right)T^{1/2}+C_1.$$

(iv) It follows from (iii) that $\{T^{1/2}A_nT^{1/2}+C_{n+1}\}$ is monotone decreasing. Since

$$\|T^{1/2}A_nT^{1/2}+C_{n+1}\|\le\|T\|\,\|A_n\|+\|C_{n+1}\|\le 2,$$

$\{T^{1/2}A_nT^{1/2}+C_{n+1}\}$ is bounded below by $-2I$ and has a strong-operator limit B in $\mathfrak{A}$. At the same time $T^{1/2}AT^{1/2}+C$ is a weak-operator limit point of $\{T^{1/2}A_nT^{1/2}+C_{n+1}\}$ where C (in $\mathfrak{A}$) is the strong-operator limit of $\{C_j\}$. Thus $T^{1/2}AT^{1/2}+C=B$ and $T^{1/2}AT^{1/2}$ $(=B-C)\in\mathfrak{A}$.

(v) Since $T\in\mathfrak{A}$, $R(T)\in\mathfrak{A}$ from Lemma 4.1 (ii). Now $\langle T(z+\sum_{k=1}^{\infty}(A_k-A_{k-1})^+z),u\rangle$ $(=\langle z+\sum_{k=1}^{\infty}(A_k-A_{k-1})^+z,Tu\rangle)$ is approximated closely by:

$$\left\langle\left(z+\sum_{k=1}^{n}(A_k-A_{k-1})^+z\right),\left[I+\sum_{k=1}^{n}(A_k-A_{k-1})^+\right]^{-1}u\right\rangle=\langle z,u\rangle$$

for large n, where u is any preassigned vector in $\mathscr{H}$ and z is either x or y. (From (i), $\sum_{k=1}^{\infty}(A_k-A_{k-1})^+z$ converges to some vector in $\mathscr{H}$ when z is either x or y.)

Thus

$$T\left(z+\sum_{k=1}^{\infty}(A_k-A_{k-1})^+ z\right)=z, \quad \text{and} \quad x, y\in R(T)(\mathscr{H}).$$

(vi) Suppose $\mathscr{A}$ is a maximal abelian self-adjoint subalgebra of $\mathfrak{A}$. If $\{A_a\}$ is a bounded increasing net in $\mathscr{A}$, its strong-operator limit lies in $\mathfrak{A}$ and commutes with $\mathscr{A}$. Hence that limit lies in $\mathscr{A}$. It follows that $\mathscr{A}$ satisfies the same condition as $\mathfrak{A}$, so that what we have proved thus far about $\mathfrak{A}$ applies to $\mathscr{A}$. In other words, for this part of the proof, we may assume that $\mathfrak{A}$ is abelian. With that assumption, $T^{1/2}AT^{1/2}=AT\in\mathfrak{A}$. Since x is in the range of T and $Ax=x$, x is in the range of AT. In addition AT $(=TA)$ is self-adjoint and $TAy=0$ since $Ay=0$. Thus $R(TA)\in\mathfrak{A}$, $R(TA)x=x$, and $R(TA)y=0$. From Lemma 4.1 (v) (vi), $\mathfrak{A}=\mathfrak{A}^-$ (that is, $\mathscr{A}=\mathscr{A}^-$) in this case.

Lemma 4.3. *With the notation and assumptions of Lemma* 4.2*:*

(i) *MAN lies in $\mathfrak{A}$, where M and N are spectral projections for T corresponding to bounded intervals with positive left endpoints;*

(ii) *M_mAF and FAM_m are in $\mathfrak{A}$, where $F=R(T)$ and $\{M_m\}$ is a sequence of spectral projections for T corresponding to bounded intervals with positive left endpoints such that $\sum_m M_m=F$;*

(iii) *$FAFAF\in\mathfrak{A}$;*

(iv) *$FAFAFx=x$ and $FAFAFy=0$;*

(v) *$\mathfrak{A}=\mathfrak{A}^-$.*

Proof. (i) Let S be a bounded interval with a positive left endpoint and let $g(t)$ be t^{-1} for t in S and O for t in $\mathbb{R}\backslash S$. From Lemma 4.2 (vi), a maximal abelian subalgebra $\mathscr{A}$ of $\mathfrak{A}$ containing T is weak-operator closed in $\mathscr{B}(\mathscr{H})$ and, therefore, contains $g(T)$. From [8; Theorem 5.2.8], $g(T)T=M$, where M is the spectral projection for T corresponding to S. Since $TAT\in\mathfrak{A}$ from Lemma 4.2 (iv), MAT $(=g(T)TAT)\in\mathfrak{A}$. Similarly, $MAN\in\mathfrak{A}$, where N is another spectral projection for T corresponding to a bounded interval with positive left endpoint.

(ii) Since

$$(M_mAM_n+M_n)(M_nAM_m+M_n)=M_mAM_nAM_m+M_mAM_n+M_nAM_m+M_n$$

and $\sum_{n=1}^{\infty}M_n$ $(=F)\in\mathfrak{A}$, we have that

$$M_mAFAM_m+M_mAF+FAM_m+F\in\mathfrak{A}$$

and also $M_mAFAM_m\in\mathfrak{A}$. (Both operators are strong-operator convergent sums of positive operators in $\mathfrak{A}$.) Thus

$$M_mAF+FAM_m\in\mathfrak{A}, \text{ and } M_mAF+M_mAM_m \ (=M_m[M_mAF+FAM_m])\in\mathfrak{A}.$$

Since $M_m A M_m \in \mathfrak{A}$, we have that $M_m AF \in \mathfrak{A}$.

(iii) From (ii),

$$(M_m AF)^*(M_m AF)\ (=FAM_m AF) \in \mathfrak{A}.$$

Again,

$$FAFAF\left(=\sum_{m=1}^{\infty} FAM_m AF\right) \in \mathfrak{A}.$$

(iv) From Lemma 4.2 (v), $Fx=x$ and $Fy=y$, so that $FAFAFx=x$ and $FAFAFy=0$.

(v) Combining the conclusions of Lemma 4.1 with what we have proved thus far, we see that for each cyclic projection E in $\mathfrak{A}^-$ with generating unit vector x and each unit vector y in $(I-E)(\mathscr{H})$, there is a self-adjoint operator $FAFAF$ in $\mathfrak{A}$ such that $FAFAFx=x$ and $FAFAFy=0$. The conditions of Lemma 4.1 (v), (vi) are fulfilled, and $\mathfrak{A}=\mathfrak{A}^-$.

Recall (Definition 2.1) that a C^*-algebra $\mathfrak{A}$ with the property that each increasing net in $\mathfrak{A}$ that is bounded above has a least upper bound in $\mathfrak{A}$ and for which there is a separating family of states whose limits on such nets is their values at the least upper bounds is called a W^*-algebra and the special states are called *normal states.*

Theorem 4.4. *A C^*-algebra $\mathfrak{A}$ is * isomorphic to a von Neumann algebra if and only if it is a* W^*-algebra.

Proof. Suppose $\mathfrak{A}$ is * isomorphic to a von Neumann algebra. The * isomorphism transforms increasing bounded nets onto such nets, least upper bounds onto least upper bounds, and normal states onto normal states. Thus $\mathfrak{A}$ is a W^*-algebra in this case.

Suppose $\mathfrak{A}$ is a W^*-algebra and $\phi=\sum_{a\in\mathbb{A}}\oplus\pi_{\eta(a)}$, where $\{\eta(a): a\in\mathbb{A}\}$ is the family of normal states of $\mathfrak{A}$. Since $\{\eta(a)\}$ is separating for $\mathfrak{A}$, ϕ is a * isomorphism. Write x_a for $x_{\eta(a)}$ and suppose $\{\phi(A_b): b\in\mathbb{B}\}$ is a bounded increasing net in $\phi(\mathfrak{A})$ with strong-operator limit B. Then $\{A_b\}$ is a bounded increasing net in $\mathfrak{A}$. By assumption $\{A_b\}$ has a least upper bound A in $\mathfrak{A}$ and $\{\eta(a)(A_b)\}$ tends to $\eta(a)(A)$ for each a in $\mathbb{A}$. Thus $\{\langle\phi(A_b)x_a,x_a\rangle\}$ tends to $\langle\phi(A)x_a,x_a\rangle$. But $\{\langle\phi(A_b)x_a,x_a\rangle\}$ tends to $\langle Bx_a,x_a\rangle$ as well. Thus $\langle(\phi(A)-B)x_a,x_a\rangle=0$ for each a in $\mathbb{A}$. With T invertible in $\mathfrak{A}$, $\{T^*A_bT\}$ has T^*AT as least upper bound and $\phi(T^*A_bT)$ has $\phi(T)^*B\phi(T)$ as strong-operator limit. Thus $\langle(\phi(A)-B)\phi(T)x_a,\phi(T)x_a\rangle=0$ for each a in $\mathbb{A}$ and each invertible T in $\mathfrak{A}$. With S in $\mathfrak{A}$, $S+nI$ is an invertible element of $\mathfrak{A}$ for all large positive integers n. Thus

$$\begin{aligned}0=&\langle(\phi(A)-B)\phi(S)x_a,\phi(S)x_a\rangle\\ &+2nRe\langle(\phi(A)-B)\phi(S)x_a,x_a\rangle+n^2\langle(\phi(A)-B)x_a,x_a\rangle\end{aligned}$$

when n is large. But $\langle(\phi(A)-B)x_a, x_a\rangle=0$ and $\langle(\phi(A)-B)\phi(S)x_a, \phi(S)x_a\rangle$ is independent of n. Thus $\langle(\phi(A)-B)\phi(S)x_a, \phi(S)x_a\rangle=0$ for each S in $\mathfrak{A}$. Since x_a is a cyclic vector for the representation $\pi_{\eta(a)}$ and ϕ is the direct sum of $\{\pi_{\eta(a)}: a\in\mathbb{A}\}$, $\phi(A)=B$. Hence $\phi(\mathfrak{A})$ satisfies the conditions of Lemma 4.3 and $\phi(\mathfrak{A})=\phi(\mathfrak{A})^{-}$. Thus $\mathfrak{A}$ is a W^*-algebra.

By studying ghe proofs of Lemmas 4.1–4.3, we note that the only use of *nets* as opposed to *sequences* is to establish that arbitrary unions of projections in $\mathfrak{A}$ lie in $\mathfrak{A}$. With this observation, we can prove the following strengthened version of Lemma 4.3 when $\mathfrak{A}$ satisfies a certain "countability" assumption (always fulfilled when the underlying Hilbert space is separable). This altered version of Lemma 4.3 can then be used to characterize countably decomposable von Neumann algebras (Theorem 4.6).

Lemma 4.5. *Let $\mathfrak{A}$ be a C^*-algebra acting on a Hilbert space. Suppose that each bounded increasing sequence in $\mathfrak{A}$ has its strong-operator limit in $\mathfrak{A}$ and that each orthogonal family of non zero projections in $\mathfrak{A}$ is countable. Then $\mathfrak{A}=\mathfrak{A}^{-}$.*

Proof. As indicated in the preceding comments, it suffices to establish that the union of an increasing net of projections in $\mathfrak{A}$ is in $\mathfrak{A}$ under the present assumptions. We show that the union F of an arbitrary family $\{F_a: a\in\mathbb{A}\}$ of projections in $\mathfrak{A}$ lies in $\mathfrak{A}$. Let $\{E_b: b\in\mathbb{B}\}$ be a maximal orthogonal family of (non-zero) projections in $\mathfrak{A}$ such that $E_b\le F$ for each b. By assumption $\mathbb{B}$ is countable (possibly finite) so that we can denote the family $\{E_b\}$ by $\{E_1, E_2, \ldots\}$. Since $\{E_1, E_1+E_2, \ldots\}$ is an increasing sequence of projections in $\mathfrak{A}$, its strong-operator limit $\sum_n E_n$ $(=E)$ is in $\mathfrak{A}$ by assumption. We assert that $E=F$. Since $E\le F$, $E\vee F_a\le F$ for each a in $\mathbb{A}$. The range projection of $E+F_a$ is $E\vee F_a$ and is the strong-operator limit of the increasing sequence $\{[(E+F_a)/2]^{1/n}\}$. Thus $E\vee F_a$ is in $\mathfrak{A}$ as is $E\vee F_a-E$. If $E\vee F_a-E\neq 0$, it can be adjoined to $\{E_1, E_2, \ldots\}$ to form a larger orthogonal family of non-zero projections in $\mathfrak{A}$ contained in F-contradicting the maximality of $\{E_1, E_2, \ldots\}$. Thus $E\vee F_a=E$, $F_a\le E$ for each a in $\mathbb{A}$, and $F=E\in\mathfrak{A}$.

Theorem 4.6. *A C^*-algebra $\mathfrak{A}$ is * isomorphic to a countably decomposable von Neumann algebra $\mathscr{R}$ if and only if each bounded increasing sequence in $\mathfrak{A}$ has a least upper bound in $\mathfrak{A}$, there is a separating family of (normal) states of $\mathfrak{A}$ whose limits on such a sequence are their values at the least upper bound, and each orthogonal family of non-zero projections in $\mathfrak{A}$ is countable.*

Proof. Suppose $\mathfrak{A}$ is * isomorphic to a countably decomposable von Neumann algebra $\mathscr{R}$ acting on a Hilbert space $\mathscr{H}$. Then bounded increasing sequences in $\mathfrak{A}$ map onto such sequences in $\mathscr{R}$ under the isomorphism. The least upper bound of the image sequence in $\mathscr{R}$ is the image of an element of $\mathfrak{A}$ that is the least upper bound of the sequence in $\mathfrak{A}$. Vector states of $\mathscr{R}$ composed with the isomorphism are normal states of $\mathfrak{A}$, and the set of such form a separating family for $\mathscr{R}$. An orthogonal family of non-zero projections in $\mathfrak{A}$ maps onto such a family in $\mathscr{R}$. Since $\mathscr{R}$ is countably decomposable, the family of projections is countable.

The argument of Theorem 4.4 applies to a C^*-algebra $\mathfrak{A}$ satisfying the given conditions, with sequences and Lemma 4.5 used in place of nets and Lemma 4.3. Thus $\mathfrak{A}$ is * isomorphic to a von Neumann algebra $\mathscr{R}$. From the condition on orthogonal families of projections in $\mathfrak{A}$, $\mathscr{R}$ is countably decomposable.

We turn next to Sakai's characterization of a von Neumann algebra as a C^*-algebra dual to some Banach space [16]. We derive it from Theorem 4.4 in a sequence of results leading to Theorem 4.11.

Lemma 4.7. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{A}_\#$ be a Banach space such that $\mathfrak{A}$ is (isometrically isomorphic to) the (norm) dual space of $\mathfrak{A}_\#$.*

(i) *An element A in $\mathfrak{A}$ is a self-adjoint element in the ball $(\mathfrak{A})_r$ of radius r in $\mathfrak{A}$ with center 0 if and only if $\|A+inI\|^2 \le r^2+n^2$ for each integer n.*

(ii) *The set of self-adjoint elements in $(\mathfrak{A})_r$ is weak * closed in $\mathfrak{A}$.*

(iii) *The set $(\mathfrak{A}^+)_r$ of positive elements in $(\mathfrak{A})_r$ is weak * closed in $\mathfrak{A}$.*

Proof. (i) Note that for each A in $\mathfrak{A}$

$$\|A+inI\|^2 = \|(A^*-inI)(A+inI)\| = \|A^*A+in(A^*-A)+n^2I\|.$$

Thus if A is a self-adjoint element in $(\mathfrak{A})_r$,

$$\|A+inI\|^2 = \|A^*A+n^2I\| = \|A\|^2+n^2 \le r^2+n^2,$$

for each integer n.

Suppose $A=A_1+iA_2$ with A_1 and A_2 self-adjoint elements in $\mathfrak{A}$. If $A_2 \neq 0$, then some non-zero b lies in $sp A_2$. Now $b+n \in sp(A_2+nI)$, so that

$$b^2+2bn+n^2=(b+n)^2 \le \|A_2+nI\|^2 \le \|A+inI\|^2.$$

But with $|n|$ large and nb positive,

$$r^2+n^2<b^2+2bn+n^2;$$

whence $r^2+n^2< \|A+inI\|^2$ for such n.

(ii) We think of the elements of $\mathfrak{A}_\#$ as linear functionals on $\mathfrak{A}$ by means of their isometric injection into the norm dual of $\mathfrak{A}$ (the second dual of $\mathfrak{A}_\#$). If η is such an element, we write $\eta(A)$ for the value of A at η. With this notation,

$$\|A\| = \sup\{|\eta(A)| : \eta \in (\mathfrak{A}_\#)_1\}.$$

Thus, for each integer n,

$$\begin{aligned}\bigcap_{\eta\in(\mathfrak{A}_\#)_1} &\{A\in\mathfrak{A} : |\eta(A+inI)| \le (r^2+n^2)^{1/2}\} \\ &= \{A\in\mathfrak{A} : \|A+inI\| \le (r^2+n^2)^{1/2}\},\end{aligned}$$

and these sets are weak * closed. From (i),

$$\bigcap_{n\in\mathbb{Z}}\{A\in\mathfrak{A}: \|A+inI\|^2\le r^2+n^2\}=(\mathfrak{A}_h)_r,$$

so that $(\mathfrak{A}_h)_r$ is weak * closed.

(iii) From [8; Lemma 4.2.1], if $\|A\|\le r$ and A is self-adjoint, A is positive if and only if $\|A-rI\|\le r$. Thus $A\in(\mathfrak{A}^+)_r$ if and only if $A\in(\mathfrak{A}_h)_r$ and $|\eta(A-rI)|\le r$ for each η in $(\mathfrak{A}_*)_1$. That is

$$(\mathfrak{A}^+)_r=\bigcap_{\eta\in(\mathfrak{A}_*)_1}\{A\in(\mathfrak{A}_h)_r: |\eta(A-rI)|\le r\}.$$

From (ii), each of the sets $\{A\in(\mathfrak{A}_h)_r: |\eta(A-rI)|\le r\}$ is weak * closed. Thus $(\mathfrak{A}^+)_r$ is weak * closed.

Lemma 4.8. *Adopt the notation of Lemma 4.7, and let $\mathscr{F}$ be the family of subsets of $\mathfrak{A}$ whose intersection with every $(\mathfrak{A})_n$ is weak * closed, where n is a positive integer. With $\mathscr{S}$ a subset of $\mathfrak{A}_*$ and a a positive number, denote by $\mathfrak{A}(\mathscr{S},a)$ and $\mathfrak{A}^0(\mathscr{S},a)$, respectively, the subsets*

$$\{A\in\mathfrak{A}: |\eta(A)|\le a, \eta\in\mathscr{S}\},\qquad \{A\in\mathfrak{A}: |\eta(A)|< a, \eta\in\mathscr{S}\};$$

let $\mathfrak{A}_n(\mathscr{S},a)$ and $\mathfrak{A}^0_n(\mathscr{S},a)$ denote the sets $\mathfrak{A}(\mathscr{S},a)\cap(\mathfrak{A})_n$ and $\mathfrak{A}^0(\mathscr{S},a)\cap(\mathfrak{A})_n$, respectively.

(i) *$\mathscr{F}$ is the family of closed sets of a topology (the "$\mathscr{F}$-topology") for $\mathfrak{A}$.*

(ii) *If $\mathcal{O}$ is an $\mathscr{F}$-open subset of $\mathfrak{A}$, then for each A in $\mathfrak{A}$, $A+\mathcal{O}$ is an $\mathscr{F}$-open set, and the mapping $B\to A+B$ of $\mathfrak{A}$ onto itself is an $\mathscr{F}$-homeomorphism.*

(iii) *If $\{\eta_j\}$ is a sequence of elements of $\mathfrak{A}_*$ tending to 0 in norm, then $\mathfrak{A}^0(\{\eta_j\},a)$ is $\mathscr{F}$-open for each positive a.*

(iv) *If $\mathcal{O}$ is an $\mathscr{F}$-open set containing 0 and n is a positive integer, there is a finite subset $\mathscr{S}$ of $\mathfrak{A}_*$ such that $\mathfrak{A}_n(\mathscr{S},1)$ is contained in $\mathcal{O}\cap(\mathfrak{A})_n$, and there is a finite subset $\mathscr{T}$ of $(\mathfrak{A}_*)_{1/n}$ such that*

$$\mathfrak{A}_{n+1}(\mathscr{S}\cup\mathscr{T},1)\subseteq\mathcal{O}\cap(\mathfrak{A})_{n+1}.$$

(v) *If $\mathcal{O}$ is an $\mathscr{F}$-open set containing 0, there is a sequence $\{\mathscr{S}_n\}$ of finite sets $\mathscr{S}_n$ such that $\mathscr{S}_{n+1}\subseteq(\mathfrak{A}_*)_{1/n}$ for n in $\{1,a,\ldots\}$ and such that for n in $\mathbb{N}$,*

$$\mathfrak{A}_n(\mathscr{S}_1\cup\ldots\cup\mathscr{S}_n,1)\subseteq\mathcal{O}\cap(\mathfrak{A})_n.$$

(vi) *For a given $\mathscr{F}$-open set $\mathcal{O}$ containing 0, there is a sequence $\{\eta_j\}$ in $\mathfrak{A}_*$ tending to 0 in norm such that $\mathfrak{A}(\{\eta_j\},1)\subseteq\mathcal{O}$.*

(vii) *Given an $\mathscr{F}$-open set $\mathcal{O}$ in $\mathfrak{A}$ containing 0, there is an $\mathscr{F}$-open set $\mathcal{O}_0$ containing 0 such that $\mathcal{O}_0+\mathcal{O}_0\subseteq\mathcal{O}$, and addition is $\mathscr{F}$-continuous on $\mathfrak{A}$.*

(viii) *The mapping*

$$(a,A)\to aA : \mathbb{C}\times\mathfrak{A}\to\mathfrak{A}$$

is $\mathscr{F}$-continuous, so that $\mathscr{F}$ provides $\mathfrak{A}$ with a locally convex linear topological structure.

Proof. (i) If $\{F_a\}$ is a subset of $\mathscr{F}$, then

$$\left(\bigcap_a F_a\right)\cap(\mathfrak{A})_n=\bigcap_a(F_a\cap(\mathfrak{A})_n)$$

and each $F_a\cap(\mathfrak{A})_n$ is weak * closed by definition of $\mathscr{F}$. Thus $\bigcap_a F_a\in\mathscr{F}$. If $\{F_1,\dots,F_n\}$ is a finite subset of $\mathscr{F}$, then

$$\left(\bigcup_{j=1}^{m} F_j\right)\cap(\mathfrak{A})_n=\bigcup_{j=1}^{m}(F_j\cap(\mathfrak{A})_n);$$

and $\bigcup_{j=1}^{m} F_j\in\mathscr{F}$. In addition, $\emptyset$ and $\mathfrak{A}$ are in $\mathscr{F}$ as is each weak * closed subset of $\mathfrak{A}$. Thus $\mathscr{F}$ is the family of closed subsets for a topology on $\mathfrak{A}$, the $\mathscr{F}$-topology.

(ii) We show that $(A+\mathcal{O})\cap(\mathfrak{A})_n$ is a (relative) weak * open subset of $(\mathfrak{A})_n$. Suppose $C=A+B$ with B in $\mathcal{O}$ and $\|C\|\le n$ (so that $C\in(A+\mathcal{O})\cap(\mathfrak{A})_n$). Then $\|B\|=\|C-A\|\le n+\|A\|\le n+m$, where $\|A\|\le m$. Thus $B\in\mathcal{O}\cap(\mathfrak{A})_{n+m}$ and $\mathcal{O}\cap(\mathfrak{A})_{n+m}$ is a (relative) weak * open subset of $(\mathfrak{A})_{n+m}$ since $\mathcal{O}$ is $\mathscr{F}$-open. It follows that there is a weak * open set $\mathcal{O}_1$ in $\mathfrak{A}$ such that $\mathcal{O}\cap(\mathfrak{A})_{n+m}=\mathcal{O}_1\cap(\mathfrak{A})_{n+m}$. Now $A+\mathcal{O}_1$ is weak * open, so that $(A+\mathcal{O}_1)\cap(\mathfrak{A})_n$ is a weak * open subset of $(\mathfrak{A})_n$ containing $C(=A+B)$. Moreover, $(A+\mathcal{O}_1)\cap(\mathfrak{A})_n\subseteq(A+\mathcal{O})\cap(\mathfrak{A})_n$. Thus $(A+\mathcal{O})\cap(\mathfrak{A})_n$ is a weak * open subset of $(\mathfrak{A})_n$ for each positive integer n, and $A+\mathcal{O}$ is $\mathscr{F}$-open. Since the mappings $B\to A+B$ and $B\to -A+B$ are inverse to each other and both are $\mathscr{F}$-open mappings of $\mathfrak{A}$ onto $\mathfrak{A}$, both mappings are $\mathscr{F}$-homeomorphisms.

(iii) We show that $\mathfrak{A}_n^0(\{\eta_j\},a)$ is a weak * open subset of $(\mathfrak{A})_n$ for each positive integer n. In fact, since $\|\eta_j\|\to 0$, there is a positive integer n_0 such that $\|\eta_j\|<a/n$ when $j>n_0$. Thus $|\eta_j(A)|\le\|\eta_j\|\,\|A\|<a$ when $A\in(\mathfrak{A})_n$ and $j>n_0$. It follows that

$$\mathfrak{A}_n^0(\{\eta_j\},a)=\mathfrak{A}_n^0(\{\eta_1,\dots,\eta_{n_0}\},a)$$

and $\mathfrak{A}_n^0(\{\eta_j\},a)$ is a weak * open subset of $(\mathfrak{A})_n$.

(iv) Since $\mathcal{O}\cap(\mathfrak{A})_n$ is a weak * open subset of $(\mathfrak{A})_n$ containing $\mathcal{O}$, there is a weak * open set $\mathcal{O}_0$ containing 0 such that $\mathcal{O}\cap(\mathfrak{A})_n=\mathcal{O}_0\cap(\mathfrak{A})_n$. Since $\mathcal{O}_0$ is weak * open, there is a finite subset $\mathscr{S}$ of elements of $\mathfrak{A}_\#$ such that $\mathfrak{A}(\mathscr{S},1)\subseteq\mathcal{O}_0$, whence

$$\mathfrak{A}_n(\mathscr{S},1)\subseteq\mathcal{O}_0\cap(\mathfrak{A})_n=\mathcal{O}\cap(\mathfrak{A})_n.$$

Suppose that for each finite subset $\mathscr{T}$ of $(\mathfrak{A}_\#)_{1/n}$,

$$\mathfrak{A}_{n+1}(\mathscr{S}\cup\mathscr{T},1)\nsubseteq\mathcal{O}\cap(\mathfrak{A})_{n+1}.$$

Then for each such $\mathscr{T}$,

$$\mathfrak{A}_{n+1}(\mathscr{S}\cup\mathscr{T},1)\cap(\mathfrak{A}\setminus\mathcal{O})\neq\emptyset.$$

Let $\mathcal{T}_1, \dots, \mathcal{T}_k$ be finite subsets of $(\mathfrak{A}_\#)_{1/n}$, and let $\mathcal{T}_0$ be their union. Then $\mathcal{T}_0$ is a finite subset of $(\mathfrak{A}_\#)_{1/n}$ and

$$\mathfrak{A}(\mathcal{S}\cup\mathcal{T}_0, 1) = \bigcap_{j=1}^{k} \mathfrak{A}(\mathcal{S}\cup\mathcal{T}_j, 1).$$

Now $\mathfrak{A}_{n+1}(\mathcal{S}\cup\mathcal{T}_0, 1)\cap(\mathfrak{A}\backslash\mathcal{O})\neq\emptyset$; and, hence; the total collection of sets $\{\mathfrak{A}_{n+1}(\mathcal{S}\cup\mathcal{T}, 1)\cap(\mathfrak{A}\backslash\mathcal{O}): \mathcal{T}$ a finite subset of $(\mathfrak{A}_\#)_{1/n}\}$ has the finite intersection property. Since each set $\mathfrak{A}(\mathcal{S}\cup\mathcal{T}, 1)$ is weak * closed as is $(\mathfrak{A}\backslash\mathcal{O})\cap(\mathfrak{A})_{n+1}$ (for $\mathfrak{A}\backslash\mathcal{O}$ is $\mathcal{F}$-closed), and $(\mathfrak{A})_{n+1}$ is weak * compact, the entire collection of sets has a non-empty intersection. Let A_0 be an element of the intersection. For each η in $(\mathfrak{A}_\#)_{1/n}$,

$$A_0\in\mathfrak{A}_{n+1}(\mathcal{S}\cup\{\eta\}, 1)\cap(\mathfrak{A}\backslash\mathcal{O}),$$

so that $|\eta(A_0)|\le 1$. Since $\mathfrak{A}$ is the dual of $\mathfrak{A}_\#$, $\|A_0\|-\varepsilon=|\eta_0(A_0)|$ for some η_0 in $(\mathfrak{A}_\#)_1$, where ε is a preassigned positive number. If $\eta=n^{-1}\eta_0$, then $\eta\in(\mathfrak{A}_\#)_{1/n}$ and

$$\|A_0\|-\varepsilon=|\eta_0(A_0)|=n|\eta(A_0)|\le n.$$

Thus $A_0\in(\mathfrak{A})_n$. By choice of A_0 and $\mathcal{S}$,

$$A_0\in\mathfrak{A}_n(\mathcal{S}\cup\{\eta\}, 1)\subseteq\mathfrak{A}_n(\mathcal{S}, 1)\subseteq\mathcal{O}\cap(\mathfrak{A})_n.$$

But $A_0\in\mathfrak{A}\backslash\mathcal{O}$ – a contradiction. Thus there is a finite subset $\mathcal{T}$ of $(\mathfrak{A}_\#)_{1/n}$ such that

$$\mathfrak{A}_{n+1}(\mathcal{S}\cup\mathcal{T}, 1)\subseteq\mathcal{O}\cap(\mathfrak{A})_{n+1}.$$

(v) From (iv), there is a finite subset $\mathcal{S}_1$ of $\mathfrak{A}_\#$ such that $\mathfrak{A}_1(\mathcal{S}_1, 1)\subseteq\mathcal{O}\cap(\mathfrak{A})_1$ and a finite subset $\mathcal{S}_2$ of $(\mathfrak{A}_\#)_1$ such that $\mathfrak{A}_2(\mathcal{S}_1\cup\mathcal{S}_2, 1)\subseteq\mathcal{O}\cap(\mathfrak{A})_2$. Similarly, there is a finite subset $\mathcal{S}_3$ of $(\mathfrak{A}_\#)_{1/a}$ such that $\mathfrak{A}_3(\mathcal{S}_1\cup\mathcal{S}_2\cup\mathcal{S}_3, 1)\subseteq\mathcal{O}\cap(\mathfrak{A})_3$. Continuing in this way, we construct the sequence, $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \dots$, with the stated properties.

(vi) The sequence $\{\eta_j\}$ constructed by enumerating the elements of the sequence constructed in (v), $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \dots$, successively, tends to 0 in norm in $\mathfrak{A}_\#$, and

$$\mathfrak{A}_n(\{\eta_j\}_{j\in\mathbb{N}}, 1)\subseteq\mathfrak{A}_n(\mathcal{S}_1\cup\dots\cup\mathcal{S}_n, 1)\subseteq\mathcal{O}\cap(\mathfrak{A})_n$$

for each positive integer n. With A in $\mathfrak{A}(\{\eta_j\}, 1)$, choose an integer n such that $\|A\|<n$. Then

$$A\in\mathfrak{A}_n(\{\eta_j\}, 1)\subseteq\mathcal{O}\cap(\mathfrak{A})_n$$

and $A\in\mathcal{O}$. Thus $\mathfrak{A}(\{\eta_j\}, 1)\subset\mathcal{O}$.

(vii) Let $\{\eta_j\}$ be the sequence constructed in (vi). From (iii), $\mathfrak{A}^0(\{\eta_j\}, 1/2)$ is an $\mathcal{F}$-open set $\mathcal{O}_0$ containing 0. If A and B are in $\mathcal{O}_0$, then $|\eta_j(A)|<1/2$ and $|\eta_j(B)|<1/2$, whence $|\eta_j(A+B)|<1$ and $A+B\in\mathfrak{A}^0(\{\eta_j\}, 1)\subseteq\mathcal{O}$. Thus $\mathcal{O}_0+\mathcal{O}_0\subseteq\mathcal{O}$.

Given A_0, B_0 in $\mathfrak{A}$ and an $\mathcal{F}$-open neighborhood $\mathcal{O}_1$ of A_0+B_0, $-(A_0+B_0)+\mathcal{O}_1$ is an $\mathcal{F}$-open neighborhood $\mathcal{O}$ of 0 from (ii). As in the preceding paragraph, find an $\mathcal{F}$-open neighborhood $\mathcal{O}_0$ of 0 such that $\mathcal{O}_0+\mathcal{O}_0\subseteq\mathcal{O}$. Again from (ii), $A^0+\mathcal{O}_0$

and $B_0+\mathcal{O}_0$ are $\mathscr{F}$-open neighborhoods of A_0 and B_0, respectively, and

$$A_0+\mathcal{O}_0+B_0+\mathcal{O}_0=A_0+B_0+\mathcal{O}_0+\mathcal{O}_0\subseteq A_0+B_0+\mathcal{O}=\mathcal{O}_1.$$

Thus addition is $\mathscr{F}$-continuous on $\mathfrak{A}$.

(viii) Given a_0 in $\mathbb{C}$, A_0 in $\mathfrak{A}$, and an $\mathscr{F}$-open neighborhood $\mathcal{O}$ of 0, we can find a sequence $\{\eta_j\}$ of elements of $\mathfrak{A}_*$ tending to 0 in norm such that $\mathfrak{A}(\{\eta_j\},1)\subseteq\mathcal{O}$ from (vi). Let b be the smallest of 1, $1/2(|a_0|+1)$, and $1/2(|\eta_j(A_0)|+1)$ (for j in $\mathbb{N}$). From (iii), $\mathfrak{A}^0(\{\eta_j\},b)$ is an $\mathscr{F}$-open neighborhood $\mathcal{O}_0$ of 0. If $|a-a_0|<b$ and $A\in A_0+\mathcal{O}_0$, then

$$\begin{aligned}|\eta_j(aA-a_0A_0)|&\le|\eta_j(aA-aA_0)|+|\eta_j(aA_0-a_0A_0)|\\&\le|a|\,|\eta_j(A-A_0)|+|a-a_0|\,|\eta_j(A_0)|\\&\le(|a_0|+1)b+b|\eta_j(A_0)|\le 1/2+1/2\le 1.\end{aligned}$$

Thus $aA\in a_0A_0+\mathcal{O}$, and the mapping

$$(a,A)\to aA\ :\ \mathbb{C}\times\mathfrak{A}\to\mathfrak{A}$$

is $\mathscr{F}$-continuous. It follows from this and (vii) that $\mathscr{F}$ imposes a linear topological structure on $\mathfrak{A}$. As noted, given the $\mathscr{F}$-open neighborhood $\mathcal{O}$ of 0, $\mathfrak{A}(\{\eta_j\},1)\subseteq\mathcal{O}$ and $\mathfrak{A}^0(\{\eta_j\},1)$ is a convex $\mathscr{F}$-open neighborhood of 0. Thus $\mathscr{F}$ imposes a locally convex topology on $\mathfrak{A}$.

Lemma 4.9. *With the notation and terminology of Lemma* 4.8;

(i) *ϱ is an $\mathscr{F}$-continuous linear functional on $\mathfrak{A}$ if and only if ϱ is weak * continuous;*

(ii) *a convex subset of $\mathfrak{A}$ is weak * closed if and only if it lies in $\mathscr{F}$;*

(iii) *the sets of self-adjoint and positive elements in $\mathfrak{A}$ are weak * closed.*

Proof. (i) If ϱ is $\mathscr{F}$-continuous and a positive ε is given, there is an $\mathscr{F}$-open neighborhood V of 0 such that $|\varrho(A)|<\varepsilon$ when $A\in V$. Now $V\cap(\mathfrak{A})_2$ is a (relative) weak * open subset of $(\mathfrak{A})_2$ containing 0. Thus ϱ is weak * continuous at 0 on $(\mathfrak{A})_2$, and by translation, ϱ is weak * continuous on $(\mathfrak{A})_1$. Hence ϱ is norm continuous on $(\mathfrak{A})_1$ and therefore bounded. Thus $\varrho\in\mathfrak{A}^*$. Again, since ϱ is weak * continuous (at 0) on $(\mathfrak{A})_1$, given a positive ε, there is a finite set of elements $\eta_1,\dots,\eta_n$ in $\mathfrak{A}_*$ such that

$$|\varrho(A)|<\varepsilon\quad\text{when}\quad A\in(\mathfrak{A})_1\quad\text{and}\quad\sum_{j=1}^{n}|\eta_j(A)|<1.$$

It follows that

$$(*)\qquad|\varrho(A)|\le\varepsilon\|A\|+\|\varrho\|\sum_{j=1}^{n}|\eta_j(A)|$$

for all A in $\mathfrak{A}$. Now $A\to\varepsilon\|A\|$ and $A\to\|\varrho\|\sum_{j=1}^{n}|\eta_j(A)|$ are semi-norms σ_1 and σ_2 on $\mathfrak{A}$. Define the semi-norm σ on the vector space $\mathfrak{A}\oplus\mathfrak{A}$ and the linear functional σ_0 on the subspace $\{(A, A): A \in \mathfrak{A}\}$ of $\mathfrak{A} \oplus \mathfrak{A}$ by

$$\sigma((A, B)) = \sigma_1(A) + \sigma_2(B), \qquad \sigma_0((A, A)) = \varrho(A).$$

Then, from (*),

$$|\sigma_0((A, A))| = |\varrho(A)| \leq \sigma_1(A) + \sigma_2(A) = \sigma((A, A)),$$

for all A, and σ_0 extends to a linear functional σ' on $\mathfrak{A} \oplus \mathfrak{A}$ satisfying

$$|\sigma'((A, B))| \leq \sigma((A, B))$$

for all A and B in $\mathfrak{A}$. Let ϱ_1 and ϱ_2 be the linear functionals on $\mathfrak{A}$ defined by

$$\varrho_1(A) = \sigma'((A, 0)), \qquad \varrho_2(A) = \sigma'((0, A)).$$

Then, for each A in $\mathfrak{A}$,

$$\varrho_1(A) + \varrho_2(A) = \sigma'((A, A)) = \sigma_0((A, A)) = \varrho(A),$$
$$|\varrho_1(A)| = |\sigma'((A, 0))| \leq \sigma((A, 0)) = \sigma_1(A) = \varepsilon \|A\|,$$
$$|\varrho_2(A)| = |\sigma'((0, A))| \leq \sigma((0, A)) = \sigma_2(A) = \|\varrho\| \sum_{j=1}^{n} |\eta_j(A)|.$$

Thus $\varrho = \varrho_1 + \varrho_2$, $\|\varrho_1\| \leq \varepsilon$, and ϱ_2 is weak * continuous on $\mathfrak{A}$ (in fact, ϱ_2 is a linear combination of $\eta_1, \ldots, \eta_n$). It follows that ϱ is a norm limit of weak * continuous linear functionals on $\mathfrak{A}$ and that ϱ is weak * continuous. (Compare [8; Theorem 1.1.7] for the appropriate formulation of the Hahn-Banach theorem and [8; Exercises 1.9.2, 1.9.14, 1.9.15] for the more general setting of the argument showing that ϱ is weak * continuous from its weak * continuity on $(\mathfrak{A})_1$.)

If ϱ is weak * continuous on $\mathfrak{A}$, then ϱ is $\mathscr{F}$-continuous since the $\mathscr{F}$-topology on $\mathfrak{A}$ is stronger than the weak * topology.

(ii) Since each weak * closed set in $\mathfrak{A}$ is $\mathscr{F}$-closed, it suffices to show that each $\mathscr{F}$-closed convex subset $\mathscr{K}$ of $\mathfrak{A}$ is weak * closed. From Lemma 4.8, the $\mathscr{F}$-topology is a locally convex topology on $\mathfrak{A}$. Thus if $A_0 \in \mathfrak{A} \setminus \mathscr{K}$, there is an $\mathscr{F}$-continuous linear functional η on $\mathfrak{A}$ and a real b such that, for each A in $\mathscr{K}$,

$$\operatorname{Re} \eta(A_0) > b \geq \operatorname{Re} \eta(A),$$

from [8; Corollary 1.2.12]. Now η is weak * continuous on $\mathfrak{A}$ from (i). Thus $\mathscr{K}$ is the intersection of the weak * closed half-spaces containing it, and $\mathscr{K}$ is weak * closed.

(iii) From Lemma 4.7, $(\mathfrak{A}^+)_n$ is weak * closed for each integer n; that is, $\mathfrak{A}^+$ is $\mathscr{F}$-closed. From (ii), $\mathfrak{A}^+$ is weak * closed since $\mathfrak{A}^+$ is convex. Similarly, the set of self-adjoint elements in $\mathfrak{A}$ is weak * closed.

Lemma 4.10. *With the notation and assumptions of Lemma* 4.8, *let* $\mathfrak{A}_{\#}^h$ *and* $\mathfrak{A}_h$ *be the real-linear spaces of hermitian elements in* $\mathfrak{A}_{\#}$ *and* $\mathfrak{A}$, *respectively.*

(i) *If* $T \in \mathfrak{A} \setminus \mathfrak{A}_h$, *there is an* η *in* $\mathfrak{A}_{\#}^h$ *such that* $\operatorname{Im} \eta(T) \neq 0$.

(ii) *If A is a non-zero element of $\mathfrak{A}_h$, there is an η in $\mathfrak{A}^h_*$ such that $\eta(A) \neq 0$, so that $\mathfrak{A}^h_*$ separates $\mathfrak{A}$.*

(iii) $\mathfrak{A}^h_* + i\mathfrak{A}^h_* = \mathfrak{A}_*$.

(iv) *If $A \in \mathfrak{A}_h \backslash \mathfrak{A}^+$, there is a state η of $\mathfrak{A}$ in $\mathfrak{A}_*$ such that $\eta(A) < 0$.*

(v) *With A and B in $\mathfrak{A}_h$, $A \leq B$ if and only if $\eta(A) \leq \eta(B)$ for each state η of $\mathfrak{A}$ in $\mathfrak{A}_*$.*

Proof. (i) From Lemma 4.9(iii), $\mathfrak{A}_h$ is closed. Since $T \in \mathfrak{A} \backslash \mathfrak{A}_h$, there is an element η' in $\mathfrak{A}_*$ and a real b such that, for each A in $\mathfrak{A}_h$,

$$\mathrm{Re}\,\eta'(A) \leq b < \mathrm{Re}\,\eta'(T).$$

For each integer n, $nA \in \mathfrak{A}_h$ so that

$$n\,\mathrm{Re}\,\eta'(A) = \mathrm{Re}\,\eta'(nA) \leq b.$$

Hence $\mathrm{Re}\,\eta'(A) = 0$ for each A in $\mathfrak{A}_h$. Let η be $i\eta'$. Then $\eta(A)$ is real for each A in $\mathfrak{A}_h$ – that is, η is a hermitian functional on $\mathfrak{A}$ in $\mathfrak{A}_*$. Since

$$\mathrm{Re}\,\eta'(0) = 0 \leq b < \mathrm{Re}\,\eta'(T),$$

we have $\mathrm{Im}\,\eta(T) = \mathrm{Im}\,i\eta'(T) > 0$.

(ii) From (i), there is a hermitian linear functional η on $\mathfrak{A}$ in $\mathfrak{A}_*$ such that $\mathrm{Im}\,\eta(iA) \neq 0$ since $iA \notin \mathfrak{A}_h$. But

$$\eta(A) = \mathrm{Re}\,\eta(A) = \mathrm{Im}\,i\eta(A) = \mathrm{Im}\,\eta(iA) \neq 0.$$

It follows that the real linear subspace $\mathfrak{A}^h_*$ separates $\mathfrak{A}$, for if $T = A_1 + iA_2$ with A_1, A_2 in $\mathfrak{A}_h$ and $T \neq 0$, then at least one of A_1, A_2 is different from 0 and there is an η in $\mathfrak{A}^h_*$ such that $\eta(T) = \eta(A_1) + i\eta(A_2) \neq 0$.

(iii) If η_0 in $\mathfrak{A}_*$ is a norm limit of elements in $\mathfrak{A}^h_*$, then $\eta_0(A) \in \mathbb{R}$ for each A in $\mathfrak{A}_h$; and $\eta_0 \in \mathfrak{A}^h_*$. Thus $\mathfrak{A}^h_*$ and $i\mathfrak{A}^h_*$ are closed, real-linear subspaces of $\mathfrak{A}_*$. Now $\mathfrak{A}^h_* + i\mathfrak{A}^h_*$ is a complex-linear subspace of $\mathfrak{A}_*$ that separates the elements of $\mathfrak{A}$. As $\mathfrak{A}$ is the dual of $\mathfrak{A}_*$, $\mathfrak{A}^h_* + i\mathfrak{A}^h_*$ is norm dense in $\mathfrak{A}_*$ from [8; Corollary 1.6.3]. Suppose η_1 and η_2 are elements of norm 1 in $\mathfrak{A}^h_*$. There is an A in $(\mathfrak{A})_1$ such that $1/2 < |\eta_1(A)|$. Multiplying A by a suitable scalar of modulus 1, we may assume that $|\eta_1(A)| = \eta_1(A) = \eta_1([A + A^*]/2)$ and, hence, that $A \in \mathfrak{A}_h$. In this case, $\eta_1(A)$ is real and $i\eta_2(A)$ is purely imaginary, so that

$$1/2 < |\eta_1(A) - i\eta_2(A)| \leq \|\eta_1 - i\eta_2\|.$$

Thus (compare [8; Exercise 1.9.5]), $\mathfrak{A}^h_* + i\mathfrak{A}^h$ is norm closed in $\mathfrak{A}_*$. Hence $\mathfrak{A}_* = \mathfrak{A}^h_* + i\mathfrak{A}^h_*$.

(iv) From Lemma 4.9(iii), $\mathfrak{A}^+$ is weak * closed. Since $A \in \mathfrak{A} \backslash \mathfrak{A}^+$, there is an η_0 in $\mathfrak{A}_*$ and a real b such that for each H in $\mathfrak{A}^+$,

$$\mathrm{Re}\,\eta_0(A) < b \leq \mathrm{Re}\,\eta_0(H).$$

In particular, with 0 for H, we have that $b \leq 0$. From (iii), $\eta_0 = \eta_1 + i\eta_2$ with η_1 and η_2 in $\mathfrak{A}_*^h$. Thus

$$\eta_1(A) = \operatorname{Re}\eta_0(A) < b \leq \operatorname{Re}\eta_0(H) = \eta_1(H)$$

since A and H are in $\mathfrak{A}_h$. Hence $\eta_1(A) < b \leq 0$. If $\eta_1(H) < 0$ for some H in $\mathfrak{A}^+$, then $n\eta_1(H) = \eta_1(nH) < b$ for a suitably large positive integral n and $nH \in \mathfrak{A}^+$. But this contradicts the property of η_1 just established. Thus $0 \leq \eta_1(H)$ for each H in $\mathfrak{A}^+$. Since $\eta_1(A) < 0$, $\eta_1 \neq 0$ and some positive scalar multiple η of η_1 is a state of $\mathfrak{A}$ in $\mathfrak{A}_*$ with the desired properties.

(v) If $A \leq B$ and η is a state of $\mathfrak{A}$, then $\eta(A) \leq \eta(B)$. On the other hand, if $A \nleq B$, then $B - A \in \mathfrak{A}_h \backslash \mathfrak{A}^+$ and $\eta(B-A) < 0$ for some state η of $\mathfrak{A}$ in $\mathfrak{A}_*$. For this state η, $\eta(B) < \eta(A)$. Thus, if $\eta(A) \leq \eta(B)$ for each state η of $\mathfrak{A}$ in $\mathfrak{A}_*$, $A \leq B$.

Theorem 4.11. *With the notation and assumptions of Lemma* 4.7,

(i) *each monotone increasing net in* $\mathfrak{A}$ *with an upper bound has a least upper bound in* $\mathfrak{A}$;

(ii) $\mathfrak{A}$ *is a* W^**-algebra.*

Proof. (i) Let $\{A_a\}_{a \in \mathbb{A}}$ be an increasing net of operators in $\mathfrak{A}$ that is bounded above. To show that $\{A_a\}$ has a least upper bound in $\mathfrak{A}$, it will suffice to show that the cofinal subnet $\{A_a\}_{a \geq a_0}$ has a least upper bound in $\mathfrak{A}$; we may assume that $\{A_a\} \subseteq (\mathfrak{A})_r$ for some positive r and that $\{A_a\}$ has a first element A_{a_0}. Since the mapping $T \rightarrow T + A$ is an order isomorphism of $\mathfrak{A}$ onto itself for each self-adjoint A in $\mathfrak{A}$, it will suffice to show that $\{A_a - A_{a_0}\}$ has a least upper bound in $\mathfrak{A}$; we may assume that $\{A_a\} \subseteq (\mathfrak{A}^+)_r$.

Since $(\mathfrak{A}^+)_r$ is weak * compact, $\{A_a\}$ has a cofinal subnet $\{A_{a'}\}$ convergent to some A in $(\mathfrak{A}^+)_r$. With η a state of $\mathfrak{A}$ in $\mathfrak{A}_*$, $\{\eta(A_{a'})\}$ converges to $\eta(A)$. As $\{A_{a'}\}$ is monotone increasing and η is a state of $\mathfrak{A}$, $\{\eta(A_{a'})\}$ is monotone increasing to $\eta(A)$. Thus $\eta(A_{a'}) \leq \eta(A)$ for each a' and each state η in $\mathfrak{A}_*$. If $A - A_a \notin \mathfrak{A}^+$ for some a in $\mathbb{A}$, then $A - A_{a'} \notin \mathfrak{A}^+$ when $a' \geq a$ and there is a state η of $\mathfrak{A}$ in $\mathfrak{A}_*$ such that $\eta(A - A_{a'}) < 0$ from Lemma 4.10(iv). But as just noted, $\eta(A_{a'}) \leq \eta(A)$ – a contradiction. Thus $A_a \leq A$ for each a in $\mathbb{A}$; A is an upper bound for $\{A_a\}$.

Suppose B is an upper bound for $\{A_a\}$ in $\mathfrak{A}$. Then $\eta(A_{a'}) \leq \eta(B)$ for each state η of $\mathfrak{A}$ and, in particular, for each such η in $\mathfrak{A}_*$. But with η in $\mathfrak{A}_*$, $\{\eta(A_{a'})\}$ converges to $\eta(A)$. Thus $\eta(A) \leq \eta(B)$ for each state η of $\mathfrak{A}$ in $\mathfrak{A}_*$. From Lemma 4.10(v), $A \leq B$; hence A is the least upper bound of $\{A_a\}$ in $\mathfrak{A}$.

(ii) From (i), each state of $\mathfrak{A}$ in $\mathfrak{A}_*$ is normal (as described in Definition 2.1). If $A \in \mathfrak{A}^+$ and $\eta(A) = 0$ for each normal state η of $\mathfrak{A}$, then $\eta(-A) \geq 0$ for each such η and $-A \geq 0$ from Lemma 4.10(v). Hence $A = 0$; the set of normal states of $\mathfrak{A}$ is separating. Combining this with the result of (i), we see that the conditions of Theorem 4.4 are fulfilled. Thus $\mathfrak{A}$ is a W^*-algebra.

5. The dual-space characterization-Tomiyama's proof

In [23], Tomiyama gives an elegant proof of Sakai's dual-space characterization by means of universal representation techniques (cf. [8; § 10.1]) and his important results on "projections of norm one" *(conditional expectations)*. We present a (modified) version of Tomiyama's proof of Sakai's characterization along with a proof of Tomiyama's theorem that a projection of norm one is a conditional expectation. (Another account of these arguments substantially identical with Tomiyama's but containing a proof of the uniqueness of the predual of a von Neumann algebra is to be found in [22; § 3.3].)

The linear mapping φ from one C^*-algebra $\mathfrak{A}$ into another C^*-algebra $\mathfrak{B}$ is said to be *positive* when $\varphi(H)\geq 0$ if $H\in\mathfrak{A}^+$. If, in addition, $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ and

$$\varphi(I)=I, \quad \varphi(BAC)=B\varphi(A)\,C$$

when $B, C\in\mathfrak{B}$ and $A\in\mathfrak{A}$, then φ is said to be a *conditional expectation* from $\mathfrak{A}$ onto $\mathfrak{B}$. Since each self-adjoint element of $\mathfrak{A}$ is the difference of two positive elements of $\mathfrak{A}$, φ maps self-adjoint elements onto self-adjoint elements and φ is adjoint preserving ("hermitian"). For each A in $\mathfrak{A}$ and B in $\mathfrak{B}$, $0\leq(A-B)^*(A-B)$ so that

$$0\leq\varphi((A-B)^*(A-B))=\varphi(A^*A)-B^*\varphi(A)-\varphi(A)^*B+B^*B.$$

Replacing B by $\varphi(A)$ in this inequality, we have

$$\varphi(A)^*\varphi(A)\leq\varphi(A^*A),$$

which holds for each A in $\mathfrak{A}$. Now $A^*A\leq\|A\|^2I$, so that $\varphi(A)^*\varphi(A)\leq\varphi(A^*A)\leq\|A\|^2I$. Thus $\|\varphi(A)\|\leq\|A\|$ and $\|\varphi\|\leq 1$. Since $\varphi(I)=I$, $\|\varphi\|=1$. By assumption, $\varphi(B)(=\varphi(B\cdot I\cdot I)=B\cdot\varphi(I)\cdot I)=B$ for each B in $\mathfrak{B}$. Thus $\varphi(\varphi(A))=\varphi(A)$, and φ is an idempotent. Hence φ is a projection of norm one mapping $\mathfrak{A}$ onto $\mathfrak{B}$. In Theorem 5.3, we shall prove the converse: each projection of norm one mapping $\mathfrak{A}$ onto $\mathfrak{B}$ is a conditional expectation from $\mathfrak{A}$ onto $\mathfrak{B}$. In the lemma that follows, the study of projections of norm one is reduced to the von Neumann algebra case with universal representation techniques.

Lemma 5.1. *Let $\mathfrak{B}$ be a C^*-subalgebra of the C^*-algebra $\mathfrak{A}$ and let φ_0 be an idempotent bounded linear mapping of $\mathfrak{A}$ onto $\mathfrak{B}$ such that $\|\varphi_0\|=1$. Suppose $\mathfrak{A}$ acting on the Hilbert space $\mathscr{H}$ is the universal representation of $\mathfrak{A}$ and $\mathfrak{A}^-$ is its weak-operator closure. Then φ_0 is a positive linear mapping of $\mathfrak{A}$ onto $\mathfrak{B}$ such that $\varphi_0(I)=I$ and φ_0 extends uniquely to an ultraweakly continuous idempotent linear mapping φ of $\mathfrak{A}^-$ onto $\mathfrak{B}^-$ such that $\|\varphi\|=1$ and φ is a positive linear mapping.*

Proof. With B in $\mathfrak{B}$, there is an A in $\mathfrak{A}$ such that $\varphi_0(A)=B$, since φ_0 maps $\mathfrak{A}$ *onto* $\mathfrak{B}$. As φ_0 is idempotent, $\varphi_0(B)=\varphi_0(\varphi_0(A))=\varphi_0(A)=B$. In particular, $\varphi_0(I)=I$, since $I\in\mathfrak{B}$. If ϱ is a state of $\mathfrak{B}$, then $(\varrho\circ\varphi_0)(I)=\varrho(I)=1$. Since $\|\varrho\circ\varphi_0\|\leq\|\varrho\|\,\|\varphi_0\|=1$, $\varrho\circ\varphi_0$ is a state of $\mathfrak{A}$ by [8; Theorem 4.3.2]. If $H\in\mathfrak{A}^+$, then $\varrho(\varphi_0(H))\geq 0$ for each state ϱ of $\mathfrak{B}$. Since $\varphi_0(H)\in\mathfrak{B}$, $\varphi_0(H)\in\mathfrak{B}^+$ by [8; Theorem 4.3.4(iii)]. Thus φ_0 is a positive linear mapping of $\mathfrak{A}$ onto $\mathfrak{B}$.

If ω is an ultraweakly continuous linear functional on $\mathfrak{B}$ (as $\mathfrak{B}$ acts on $\mathscr{H}$), then $\omega\circ\varphi_0$ is a bounded linear functional on $\mathfrak{A}$, and by [8; Proposition 10.1.1], $\omega\circ\varphi_0$ is ultraweakly continuous on $\mathfrak{A}$. Thus φ_0 is ultraweakly continuous and extends uniquely to an ultraweakly continuous linear mapping φ of $\mathfrak{A}^-$ into $\mathfrak{B}^-$ such that $\|\varphi\|=\|\varphi_0\|=1$. Since $\mathfrak{B}^-\subseteq\mathfrak{A}^-$, $\varphi\circ\varphi$ is defined, ultraweakly continuous, and coincides on $\mathfrak{A}$ with $\varphi_0\circ\varphi_0(=\varphi_0=\varphi\,|\,\mathfrak{A})$. The ultraweakly continuous mappings $\varphi\circ\varphi$ and φ agree on the ultraweakly dense subset $\mathfrak{A}$ of $\mathfrak{A}^-$ so that they agree on $\mathfrak{A}^-$. Hence φ is an idempotent. Since the unit ball of $\mathfrak{B}$ is contained in the unit ball of $\mathfrak{A}$ and $\|\varphi\|=1, \varphi$ maps the ultraweakly compact unit ball of $\mathfrak{A}^-$ onto an ultraweakly compact (hence closed) subset of $\mathfrak{B}^-$ that contains $(\mathfrak{B})_1$. From the Kaplansky density theorem, $(\mathfrak{B})_1^-=(\mathfrak{B}^-)_1$. Hence $\varphi(\mathfrak{A}^-)=\mathfrak{B}^-$. By the first paragraph of this argument, φ is a positive linear mapping.

In the next lemma, we make use of the concept of "definite state" to help us establish that the mapping φ is a conditional expectation. A state ϱ of a C^*-algebra $\mathfrak{A}$ is said to be definite on a self-adjoint element A of $\mathfrak{A}$ when $\varrho(A^2)=\varrho(A)^2$. In this case, $A-\varrho(A)I$ is in the kernels of ϱ, for $\varrho((A-\varrho(A)I)^2)=\varrho(A^2)-2\varrho(A)^2+\varrho(A)^2=0$. Thus $0=\varrho(B(A-\varrho(A)I))=\varrho((A-\varrho(A)I)B)$, and $\varrho(BA)=\varrho(B)\varrho(A)=\varrho(AB)$ for each B in $\mathfrak{A}$. We will also make use of the fact

$$(^{**})\qquad \|ET(I-E)+(I-E)SE\|=\max\{\|ET(I-E)\|,\|(I-E)SE\|\}$$

for all T and S in $\mathscr{B}(\mathscr{H})$ and each projection E. To see this, let x be a unit vector in $\mathscr{H}$. Then

$$\begin{aligned}\|ET(I-E)x+(I-E)SEx\|^2&=\|ET(I-E)x\|^2+\|(I-E)SEx\|^2\\&\le\|ET(I-E)\|^2\|(I-E)x\|^2+\|(I-E)SE\|^2\|Ex\|^2\\&\le\max\{\|ET(I-E)\|^2,\|(I-E)SE\|^2\},\end{aligned}$$

since $\|(I-E)x\|^2+\|Ex\|^2=1$. On the other hand,

$$\begin{aligned}\|ET(I-E)\|&=\sup\{\|ET(I-E)y\|:\|y\|\le1\}\\&=\sup\{\|ET(I-E)z\|:z=(I-E)y,\ \|y\|\le1\}\\&=\sup\{\|[ET(I-E)+(I-E)SE]z\|:z\in(I-E)(\mathscr{H}),\|z\|\le1\}\\&\le\|ET(I-E)+(I-E)SE\|.\end{aligned}$$

Similarly, $\|(I-E)SE\|\le\|ET(I-E)+(I-E)SE\|$, from which $(^{**})$ follows.

Lemma 5.2. *With the notation and assumptions of Lemma 5.1, let E be a projection in $\mathfrak{B}^-$ and x be a unit vector either in $E(\mathscr{H})$ or in $(I-E)(\mathscr{H})$. Then:*

(i) *$\omega_x\circ\varphi$ is a state of $\mathfrak{A}^-$ definite on E;*

(ii) $E\varphi(EA)E=E\varphi(AE)E=E\varphi(A)E,\ E\varphi(EAE)E=E\varphi(A)E,$

and $(I-E)\varphi(EA)(I-E)=(I-E)\varphi(AE)(I-E)=0$ *for each* A *in* $\mathfrak{A}^-$;

(iii) $\varphi(EAE)=E\varphi(A)E$ *for each* A *in* $\mathfrak{A}^-$;

(iv) $\varphi(EA(I-E))=(I-E)\varphi(EA(I-E))E+E\varphi(EA(I-E))(I-E)$ *for each* A *in* $\mathfrak{A}^-$;

(v) $(I-E)\varphi(EA(I-E))E=0$;

(vi) $\varphi(EA)=E\varphi(A)$ *and* $\varphi(AE)=\varphi(A)E$ *for each* A *in* $\mathfrak{A}^-$.

Proof. (i) Since $\varphi(I)=\varphi_0(I)=I$, $(\omega_x\circ\varphi)(I)=1$. From Lemma 5.1, φ is a positive linear mapping of $\mathfrak{A}^-$ onto $\mathfrak{B}^-$ so that $\omega_x\circ\varphi$ is a state of $\mathfrak{A}^-$. As $E^2=E$, the states ϱ of $\mathfrak{A}^-$ that are definite on E are those such that $\varrho(E)=\varrho(E^2)=\varrho(E)^2$; that is, the states definite on E are precisely those that take the value 1 or 0 at E. Since $E\in\mathfrak{B}^-$ and φ is idempotent with range $\mathfrak{B}^-$, $(\omega_x\circ\varphi)(E)=\omega_x(E)$. When $x\in(I-E)(\mathscr{H})$, $(\omega_x\circ\varphi)(E)=0$, and when $x\in E(\mathscr{H})$, $(\omega_x\circ\varphi)(E)=1$. Thus $\omega_x\circ\varphi$ is definite on E when x is a unit vector in either $E(\mathscr{H})$ or $(I-E)(\mathscr{H})$.

(ii) From (i) and the discussion preceding this lemma, when x is a unit vector in $E(\mathscr{H})$ or in $(I-E)(\mathscr{H})$,

$$(\omega_x\circ\varphi)(EA)=(\omega_x\circ\varphi)(E)(\omega_x\circ\varphi)(A)=\omega_x(E)(\omega_x\circ\varphi)(A)$$

for all A in $\mathfrak{A}^-$. Thus, with x a unit vector in $E(\mathscr{H})$, $\langle\varphi(EA)x,x\rangle=\langle\varphi(A)x,x\rangle$. This same equality holds for all x in $E(\mathscr{H})$, so that $E\varphi(EA)E=E\varphi(A)E$.

With x a unit vector in $(I-E)(\mathscr{H})$, we have $\langle\varphi(EA)x,x\rangle=0$. This same equality holds for all x in $(I-E)(\mathscr{H})$, so that $(I-E)\varphi(EA)(I-E)=0$. In the same way, we have that $E\varphi(AE)E=E\varphi(A)E$ and $(I-E)\varphi(AE)(I-E)=0$ for all A in $\mathfrak{A}^-$. Thus

$$E\varphi(EAE)E=E\varphi(AE)E=E\varphi(A)E.$$

(iii) Since φ is a positive linear mapping and $-\|A\|E\le EAE\le\|A\|E$, we have that

$$-\|A\|E=-\|A\|\varphi(E)\le\varphi(EAE)\le\|A\|\varphi(E)=\|A\|E.$$

Hence $\varphi(EAE)=E\varphi(EAE)E=E\varphi(A)E$ from (ii).

(iv) From (iii),

$$\begin{aligned}&\varphi(EA(I-E))\\&=E\varphi(EA(I-E))E+(I-E)\varphi(EA(I-E))E+E\varphi(EA(I-E))(I-E)\\&\qquad\qquad+(I-E)\varphi(EA(I-E))(I-E)\\&=(I-E)\varphi(EA(I-E))E+E\varphi(EA(I-E))(I-E),\end{aligned}$$

for each A in $\mathfrak{A}^-$.

(v) Suppose $(I-E)\varphi(EA(I-E))E\neq0$. Then for all large positive integers n,

$$\|E\varphi(EA(I-E))(I-E)\|\le\|n(I-E)\varphi(EA(I-E))E\|,$$

so that from the comments preceding this lemma and (iv) and since φ is an idempotent with range $\mathfrak{B}^-$ and norm not exceeding 1,

$$\begin{aligned}
&n\|(I-E)\varphi(EA(I-E))E\|\\
&=\max\{\|n(I-E)\varphi(EA(I-E))E\|, \|E\varphi(EA(I-E))(I-E)\|\}\\
&=\|E\varphi(EA(I-E))(I-E)+n(I-E)\varphi(EA(I-E))E\|\\
&=\|E\varphi(EA(I-E))(I-E)+(I-E)\varphi(EA(I-E))E\\
&\qquad\qquad\qquad\qquad+(n-1)(I-E)\varphi(EA(I-E))E\|\\
&=\|\varphi[EA(I-E)+(n-1)(I-E)\varphi(EA(I-E))E]\|\\
&\le\|EA(I-E)+(n-1)(I-E)\varphi(EA(I-E))E\|\\
&=(n-1)\|(I-E)\varphi(EA(I-E))E\|,
\end{aligned}$$

a contradiction. Thus $(I-E)\varphi(EA(I-E))E=0$.

(vi) From (iv) and (v),

$$\varphi(EA(I-E))=E\varphi(EA(I-E))(I-E).$$

Thus for each A in $\mathfrak{A}^-$, from (iii) and this last equality, we have that

$$\begin{aligned}
\varphi(A)&=\varphi(EAE)+\varphi(EA(I-E))+\varphi((I-E)AE)+\varphi((I-E)A(I-E))\\
&=E\varphi(EAE)E+E\varphi(EA(I-E))(I-E)+(I-E)\varphi((I-E)AE)E\\
&\qquad\qquad\qquad\qquad+(I-E)\varphi((I-E)A(I-E))(I-E)
\end{aligned}$$

so that

$$\begin{aligned}
E\varphi(A)&=E\varphi(EAE)E+E\varphi(EA(I-E))(I-E)\\
&=\varphi(EAE)+\varphi(EA(I-E))=\varphi(EA).
\end{aligned}$$

Similarly, $\varphi(AE)=\varphi(A)E$ for each A in $\mathfrak{A}^-$.

Theorem 5.3. *With the notation and assumptions of Lemma 5.1, φ_0, φ are conditional expectations from $\mathfrak{A}$, $\mathfrak{A}^-$ onto $\mathfrak{B}$, $\mathfrak{B}^-$, respectively.*

Proof. From Lemma 5.1, φ is a positive linear mapping of $\mathfrak{A}^-$ onto $\mathfrak{B}^-$ and $\varphi(I)=I$. Since φ_0 maps $\mathfrak{A}$ onto $\mathfrak{B}$ and is the restriction of φ to $\mathfrak{A}$, it will follow that φ_0 is a conditional expectation from $\mathfrak{A}$ onto $\mathfrak{B}$, when we establish that φ is a conditional expectation from $\mathfrak{A}^-$ onto $\mathfrak{B}^-$. For this last, it remains to show that $\varphi(BA)=B\varphi(A)$ and $\varphi(AB)=\varphi(A)B$ for each A in $\mathfrak{A}^-$ and B in $\mathfrak{B}^-$. From [8; Theorem 5.2.2(v)], given a positive ε, there is a (finite) orthogonal family $\{E_1,\ldots,E_n\}$ of projections in $\mathfrak{B}^-$ and (real) scalars $a_1,\ldots,a_n$ such that $\|B-\sum_{j=1}^n a_jE_j\|<\varepsilon/2\|A\|$, where B is a given self-adjoint element in $\mathfrak{B}^-$ and A is in $\mathfrak{A}^-$. From Lemma 5.2(vi), we have

$$\|\varphi(BA)-B\varphi(A)\|$$
$$\leq\left\|\varphi(BA)-\varphi\left(\left(\sum_{j=1}^{n}a_jE_j\right)A\right)\right\|+\left\|\varphi\left(\left(\sum_{j=1}^{n}a_jE_j\right)A\right)-B\varphi(A)\right\|$$
$$\leq\left\|BA-\left(\sum_{j=1}^{n}a_jE_j\right)A\right\|+\left\|\left(\sum_{j=1}^{n}a_jE_j\right)\varphi(A)-B\varphi(A)\right\|<\varepsilon.$$

Thus $\varphi(BA)=B\varphi(A)$ and similarly, $\varphi(AB)=\varphi(A)B$.

Theorem 5.4. *Suppose the C^*-algebra $\mathfrak{A}$ is (linearly isomorphic and isometric to) the norm dual of a Banach space $\mathfrak{A}_*$, η is the natural injection of $\mathfrak{A}_*$ into $\mathfrak{A}^*$, and $\mathfrak{A}$ acting on $\mathscr{H}$ is the universal representation of $\mathfrak{A}$.*

(i) *If v is an element of $\mathfrak{A}^{**}$, then $v\circ\eta=A$ for a unique A in $\mathfrak{A}$ (viewed as linear functionals on $\mathfrak{A}_*$).*

(ii) *If $A\in\mathfrak{A}^-$ and $\varphi(A)$ is the element of $\mathfrak{A}$ (obtained in* (i)*) such that $\hat{A}\circ\eta=\varphi(A)$, where $A\to\hat{A}$ is the natural (isometric linear) isomorphism between $\mathfrak{A}^-$ and $\mathfrak{A}^{**}$ (cf.* [8; *Proposition* 10.1.21]*), then φ is a conditional expectation from $\mathfrak{A}^-$ onto $\mathfrak{A}$.*

(iii) *If $\mathscr{K}=\varphi^{-1}(0)$, then $\mathscr{K}$ is a weak-operator-closed two-sided ideal in $\mathfrak{A}^-$ and $\mathscr{K}=\mathfrak{A}^-P$ for some central projection P in $\mathfrak{A}^-$.*

(iv) $\mathfrak{A}^-(I-P)=\mathfrak{A}(I-P)$.

(v) *$\mathfrak{A}$ is *isomorphic to $\mathfrak{A}(I-P)$, so that $\mathfrak{A}$ is a W^*-algebra.*

Proof. (i) Suppose $\xi\in(\mathfrak{A}_*)_1$. Then, since η is an isometry, $\|(v\circ\eta)(\xi)\|\leq\|v\|\,\|\eta(\xi)\|\leq\|v\|$, and $v\circ\eta$ is a bounded linear functional on $\mathfrak{A}_*$. By assumption, $\mathfrak{A}$ is the norm dual of $\mathfrak{A}_*$. Thus there is an A in $\mathfrak{A}$ such that $v\circ\eta=A$, and A is unique.

(ii) Let A be an element of $\mathfrak{A}$ (in $\mathfrak{A}^-$). We show that $\varphi(A)=A$. Since φ is a linear mapping of $\mathfrak{A}^-$ into $\mathfrak{A}$, this will show that φ is an idempotent mapping of $\mathfrak{A}^-$ onto $\mathfrak{A}$. With ξ in $\mathfrak{A}_*$,

$$\varphi(A)(\xi)=(\hat{A}\circ\eta)(\xi)=\eta(\xi)(A)=A(\xi).$$

Thus $\varphi(A)=A$. At the same time, if $B\in(\mathfrak{A}^-)_1$, then $\hat{B}\in(\mathfrak{A}^{**})_1$ and

$$\|\varphi(B)(\xi)\|=\|(\hat{B}\circ\eta)(\xi)\|\leq\|\eta(\xi)\|=\|\xi\|.$$

Thus $\|\varphi(B)\|\leq 1$. It follows that $\|\varphi\|\leq 1$, and from Theorem 5.3, φ is a conditional expectation from $\mathfrak{A}^-$ onto $\mathfrak{A}$.

(iii) We note first that $\mathscr{K}$ is weak-operator closed. We have that $A\in\mathscr{K}$ if and only if $(\hat{A}\circ\eta)(\xi)=0$ for all ξ in $\mathfrak{A}_*$. Now $\eta(\xi)\in\mathfrak{A}^*$ and $\mathfrak{A}$ acting on $\mathscr{H}$ is the universal representation of $\mathfrak{A}$, so that there are vectors $x(\xi)$ and $y(\xi)$ in $\mathscr{H}$ such that $\eta(\xi)=\omega_{x(\xi),y(\xi)}|\mathfrak{A}$. Thus $A\in\mathscr{K}$ if and only if $\omega_{x(\xi),y(\xi)}(A)=0$ for all ξ in $\mathfrak{A}_*$. It follows that $\mathscr{K}$ is weak-operator closed.

Since φ is a conditional expectation from $\mathfrak{A}^-$ onto $\mathfrak{A}$ $\varphi(BAC)=B\varphi(A)C$ for each A in $\mathfrak{A}^-$ and B, C in $\mathfrak{A}$. Thus, if $A\in\mathscr{K}$, $0=B\varphi(A)C=\varphi(BAC)$, and $BAC\in\mathscr{K}$. By weak-operator continuity of left (and then right) multiplication, $BAC\in\mathscr{K}$ when $A\in\mathscr{K}$ and $B, C\in\mathfrak{A}^-$. Hence $\mathscr{K}$ is a weak-operator-closed two-sided ideal in $\mathfrak{A}^-$. From [8; Theorem 6.8.8], there is a central projection P in $\mathfrak{A}^-$ such that $\mathscr{K}=\mathfrak{A}^-P$.

(iv) Since φ is idempotent, $A-\varphi(A)\in\mathscr{K}$ for each A in $\mathfrak{A}^-$. Thus $A-\varphi(A)\in\mathfrak{A}^-P$ and $A-\varphi(A)=[A-\varphi(A)]P$. It follows that $A(I-P)=\varphi(A)(I-P)\in\mathfrak{A}(I-P)$. Hence $\mathfrak{A}^-(I-P)=\mathfrak{A}(I-P)$.

(v) If $A\in\mathfrak{A}$ and $0\neq A(=\varphi(A))$, then $A\notin\mathscr{K}$ so that $A\notin\mathfrak{A}^-P$. Thus $A\neq AP$ and $A(I-P)\neq 0$. Since P commutes with $\mathfrak{A}$, the mapping $A\rightarrow A(I-P)$ of $\mathfrak{A}$ onto $\mathfrak{A}(I-P)$ is a *homomorphism and from the foregoing, this mapping is a *isomorphism. From (iv), $\mathfrak{A}(I-P)=\mathfrak{A}^-(I-P)$, so that $\mathfrak{A}$ is *isomorphic to the von Neumann algebra $\mathfrak{A}^-(I-P)$ (acting on $(I-P)(\mathscr{H})$). Hence $\mathfrak{A}$ is a W^*-algebra.

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